# Expectation Values and Vacuum Currents of Quantum Fields[*]

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**Summary.** Theory of expectation values is presented as an alternative to S-matrix theory for quantum fields. This change of emphasis is conditioned by a transition from the accelerator physics to astrophysics and cosmology. The issues discussed are the time-loop formalism, the Schwinger–Keldysh diagrams, the effective action, the vacuum currents, and the effect of particle creation.

## Introduction

High-energy physics will probably have to undergo major changes. The accelerators will cease being its experimental base, and it will become a part of astrophysics. Simultaneously, the S-matrix will cease being the central object of high-energy theory because the emphasis on this object is entirely owing to the accelerator setting of the problem. If there is a background radiation that originates from some initial state in the past, then where is the S-matrix here? Astrophysics and cosmology offer the evolution problems rather than the scattering problems. The gravitational collapse is a typical initial-value problem. It is such by its physical setting irrespective of whether the state of the system is classical or quantum. The nature of measurement also changes. No final state is prepared. One measures observables like temperatures or mechanical deflections and subjects these measurements to a statistical treatment to obtain the value of the observable. This means that one measures expectation values in the given initial state. S-matrix theory should give way to expectation-value theory.

There is a proof that accelerator physics is dead: Gabriele Veneziano is leaving CERN for Collège de France. At this historic moment, my mission is to convert him into a new faith. The present preaching consists of 4 lectures:

1. Formal aspects of expectation-value theory.
2. The in-vacuum state and Schwinger–Keldysh diagrams.

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3. The effective action.
4. Vacuum currents and the effect of particle creation.

Literature to Lectures 1 and 2 is in [1]–[16]. Additional literature to Lecture 3 is in [17]–[41] and to Lecture 4 in [42]–[56].

# 1 Formal Aspects of Expectation-Value Theory

**Vocabulary**

In these lectures,

$$\hat{\varphi}^i \tag{1.1}$$

denotes the quantum field. It is an operator function on a given differentiable manifold (referred to below as the base manifold), and $i$ is a point of this manifold. Generally, $\hat{\varphi}^i$ is a collection of fields, and then $i$ is a set containing also the indices labelling these fields. The hat designates an operator. The $\hat{\varphi}^i$ is an operator in a Hilbert space which is not granted. The workers have to build it with their own hands as a representation of the algebra of $\hat{\varphi}$'s. For simplicity, $\hat{\varphi}^i$ will be assumed boson and real (self-adjoint) but otherwise arbitrary.

The starting point is an operator equation for $\hat{\varphi}^i$

$$S_i(\hat{\varphi}) + J_i = 0 \tag{1.2}$$

which is understood as an expansion. It is meant that there is a c-number function $S_i(\varphi)$ understood as a collection of its Taylor coefficients at some c-number point of configuration space:

$$S_i(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} S_{ij_1\cdots j_n}(c)(\varphi - c)^{j_1} \ldots (\varphi - c)^{j_n} \ , \tag{1.3}$$

and one replaces $\varphi^j$ in this expansion with an operator. Which c-number field $c^j$ will be used for this expansion does not matter because it will always sum with the operator $(\hat{\varphi} - c)^j$ to make the full quantum field. The expansion point $c^j$ is often called "background field", and there has been much emphasis on it. In fact it is completely immaterial. I shall never make this expansion explicitly but I shall keep explicit the c-number term of the equation: a source $J_i$.

Important are only the following three points.

(1) The function $S_i(\varphi)$ is local, i.e., it depends only on $\varphi$ and its finite-order derivatives at the point $i$.
(2) The function $S_i(\varphi)$ is a gradient:

$$S_i(\varphi) = \frac{\delta}{\delta \varphi^i} S(\varphi) \ , \tag{1.4}$$

i.e., there exists an action $S(\varphi)$ generating the operator field equations. For its derivatives the following notation will be used:

$$S_{i_1 \cdots i_n}(\varphi) = \frac{\delta}{\delta \varphi^{i_1}} \cdots \frac{\delta}{\delta \varphi^{i_n}} S(\varphi) \ . \tag{1.5}$$

Of course, only the total action matters:

$$S_{\text{tot}} = S(\varphi) + \varphi^i J_i \ . \tag{1.6}$$

(3) There is a special condition on the matrix of second derivatives of $S(\varphi)$. I shall refer to this continuous matrix as $S_2$:

$$S_{ij}(\varphi) \equiv S_2(\varphi) \ . \tag{1.7}$$

By locality, $S_2$ is the kernel of some differential operator on the base manifold for which I shall use the same notation $S_2$. It is required that $S_2$ admit a well-posed Cauchy problem in which case it has the unique advanced and retarded inverses (Green's functions) $G^+$ and $G^-$:

$$S_{ij} G^{\pm jk} = -\delta_i^k \ , \qquad G^{+jk} = G^{-kj} \ . \tag{1.8}$$

Because $S_2$ is symmetric, the advanced inverse is the transpose of retarded.

One may think of $S_2$ as of a second-order hyperbolic operator which it will in fact be below but the scheme is more general. It is formalism-insensitive. One's field equations may have the second-order differential form or the first-order differential form, – the scheme will work anyway. The importance of the operator $S_2$ is in the fact that it determines the linear term of the field equations and, therefore, governs the iteration procedures. Commute $\hat{\varphi}^i$ with the field equations. Obtained will be a linear homogeneous equation for the commutator $[\hat{\varphi}^i, \hat{\varphi}^j]$. Consider the respective inhomogeneous equation and its two iterative solutions: one with the advanced inverse for $S_2$ and the other one with retarded. The equation for the commutator is solved by their difference:

$$[\hat{\varphi}^i, \hat{\varphi}^j] = \mathrm{i}\hbar \left( G^{+ij}(c) - G^{-ij}(c) \right) + O(\hat{\varphi} - c) \ . \tag{1.9}$$

In this way the algebra of $\hat{\varphi}$'s is built as an operator expansion. This is the quantization postulate.

By the setting of its Cauchy problem, the operator $S_2$ introduces the concept of causality. If $S_2$ is a second-order hyperbolic operator, this is the usual relativistic causality. But in any case the base manifold will be foliated with the Cauchy surfaces of the operator $S_2$. They will be denoted as $\Sigma$.

A function of $\hat{\varphi}$ that involves $\hat{\varphi}$ on only one Cauchy surface

$$Q(\hat{\varphi}) = Q(\hat{\varphi}\big|_{\Sigma}) \tag{1.10}$$

will be called local observable. A state defined as an eigenstate of local observables

$$Q(\hat{\varphi}\big|_{\Sigma})| \ \rangle = q| \ \rangle \tag{1.11}$$

will be called local state. This latter name may be confusing because the state is, of course, a global concept, and I am using the Heisenberg picture. But the local state is *associated* with a given $\Sigma$:

$$| \ \rangle = |\Sigma, q\rangle \ . \tag{1.12}$$

Of course, for it to be defined, one needs a complete set of commuting local observables. I call the $Q$'s observables but they may not even be Hermitian. And I shall consider them linear in $\hat{\varphi}$. If they are nonlinear, I shall make a local reparametrization of the field variables so as to make them linear.

In fact, if one has a complete set of commuting local observables, one has already built a Hilbert space. A linear combination

$$|\Sigma\rangle = \int dq \, \Psi(q)|\Sigma, q\rangle \tag{1.13}$$

is also a local state associated with $\Sigma$ provided that the function $\Psi(q)$ is external, i.e., independent of the quantum field $\hat{\varphi}^i$.

Our goal is to learn how to calculate expectation values of field observables in a local state, and I shall concentrate on the expectation value

$$\langle \Sigma | \hat{\varphi}^i | \Sigma \rangle \ . \tag{1.14}$$

However, we shall save the effort if we consider another problem first. Namely, let us recall what would we do in the case of two local states associated with different Cauchy surfaces:

$$|\Sigma_1, q_1\rangle = |1\rangle \ , \qquad |\Sigma_2, q_2\rangle = |2\rangle \ , \tag{1.15}$$

$$\Sigma_2 > \Sigma_1 \ .$$

Here and below, "greater" is a notation for "later".

### The Quantum Boundary-Value Problem

In the problem where given are two local states (1.15), the field's expectation value is replaced with the scalar product

$$\frac{\langle 2|\hat{\varphi}|1\rangle}{\langle 2|1\rangle} \stackrel{\text{def}}{=} \langle \varphi \rangle \tag{1.16}$$

which I shall call mean field although it is not mean in any state.

If our goal was the scalar product (1.16), we would use the Schwinger principle

$$\delta\langle 2|1\rangle = \mathrm{i}\langle 2|\delta S_{\text{tot}}|1\rangle \text{ or zero} \tag{1.17}$$

whose meaning is this. Consider a variation in the Taylor coefficients of the field equations, i.e., in the functional form of the total action. The solution for $\hat{\varphi}^i$ will respond and will induce a change in the functions $Q(\hat{\varphi})$ which will induce a change in their eigenstates, and finally there will be a change in the amplitude $\langle 2|1 \rangle$ induced by a change in the action. The Taylor coefficients are local. They can be varied in the region between $\Sigma_1$ and $\Sigma_2$ or outside this region. The Schwinger principle (1.17) says that, if they are varied outside, the variation of the amplitude is zero. Otherwise, this variation is expressed through the variation of the action by (1.17).

The Schwinger principle is a consequence of the commutation relations but it can also be taken for the first principle because one does not need anything else. For many purposes (but not all) it suffices to use a specific case of (1.17): a freedom of varying the source $J$. The result of this use is

$$\frac{\delta}{\delta \mathrm{i} J_{j_1}} \cdots \frac{\delta}{\delta \mathrm{i} J_{j_n}} \langle 2|1 \rangle = \begin{cases} \langle 2| \overleftarrow{T} \left( \hat{\varphi}^{j_1} \ldots \hat{\varphi}^{j_n} \right) |1 \rangle, & \text{if } \Sigma_2 > j_1, \ldots j_n > \Sigma_1 , \\ 0, & \text{otherwise} . \end{cases}$$
(1.18)

Here $T$ orders the operators $\hat{\varphi}^k$, $k \in \Sigma_k$, chronologically, i.e., places them in the order of following of their $\Sigma_k$, and the arrow over $T$ points the direction of growth of the time $\Sigma$.

Let us come back to the operator field equations. Since all $\hat{\varphi}$'s in these equations are at the same point, one can formally insert in (1.2) the sign of chronological ordering:

$$\overleftarrow{T} S_i(\hat{\varphi}) + J_i = 0 .$$
(1.19)

One may worry about additional terms in (1.19) stemming from the distinction between the chronological and ordinary operator products, and the noncommutativity of $\overleftarrow{T}$ with the derivatives in the Taylor coefficients of the equations. Because the operators in the products are at the same point, these terms are ambiguous expressions whose handling depends on the formalisms and procedures used. There is always a happy end: these terms cancel and help to cancel similar terms appearing in the subsequent calculations. Therefore, it makes sense to use such formalisms and procedures that these terms do not appear at all. This is the approach that I shall follow.

Sandwiching the equation (1.19) between the states $\langle 2|$ and $|1 \rangle$, and using (1.18), one obtains the following equation for the amplitude:

$$\left( S_i \left( \frac{\delta}{\delta \mathrm{i} J} \right) + J_i \right) \langle 2|1 \rangle = 0 .$$
(1.20)

Multiply it from the left with $\langle 2|1 \rangle^{-1}$ and pull the factors $\langle 2|1 \rangle$ in the argument of $S_i$ using the fact that this is a unitary transformation:

$$\left( S_i \left( \langle 2|1 \rangle^{-1} \frac{\delta}{\delta \mathrm{i} J} \langle 2|1 \rangle \right) + J_i \right) 1 = 0 .$$
(1.21)

In the argument, commute the operators:

$$\left( S_i \left( \frac{\delta \ln\langle 2|1\rangle}{\delta \mathrm{i} J} + \frac{\delta}{\delta \mathrm{i} J} \right) + J_i \right) 1 = 0 \tag{1.22}$$

and use that by (1.18)

$$\frac{\delta \ln\langle 2|1\rangle}{\delta \mathrm{i} J_k} = \langle \varphi^k \rangle \,. \tag{1.23}$$

The result is the following equation for the mean field:

$$\left( S_i \left( \langle \varphi \rangle + \frac{\delta}{\delta \mathrm{i} J} \right) + J_i \right) 1 = 0 \,. \tag{1.24}$$

Equation (1.24) differs from the classical field equation by the operator addition $\delta/\delta \mathrm{i} J$ to $\langle \varphi \rangle$. When this operator addition acts on 1, its effect is zero, but it will act also on $\langle \varphi \rangle$ because the summands $\langle \varphi \rangle$ and $\delta/\delta \mathrm{i} J$ do not commute. Where in (1.24) is the Planck constant? It is easy to see by dimension that $\hbar$ is just in front of $\delta/\delta \mathrm{i} J$. Therefore, if one wants to expand the equations in $\hbar$, one should expand them in $\delta/\delta \mathrm{i} J$.

The problem boils down to expanding a function $f(A + B)$ in $B$ when $A$ and $B$ do not commute. It suffices to expand the exponential function since one can write

$$f(A + B) = f\left( \frac{d}{dx} \right) \mathrm{e}^{(A+B)x} \Big|_{x=0} \tag{1.25}$$

or, equivalently,

$$f(A + B) = \mathrm{e}^{(A+B)d/dx} f(x) \Big|_{x=0} \,. \tag{1.26}$$

For the exponential function one has the identity

$$\mathrm{e}^{(A+B)x} = \mathrm{e}^{Ax} \left( 1 + \int_0^x dy \, \mathrm{e}^{-Ay} B \mathrm{e}^{(A+B)y} \right) \tag{1.27}$$

which makes the expansion possible. This all works well if the series of commutators

$$\mathrm{e}^{-A} B \mathrm{e}^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \cdots \tag{1.28}$$

terminates somewhere as in our case. Indeed, if $\langle \varphi \rangle = A$ and $\delta/\delta \mathrm{i} J = B$, then

$$[[B, A], A] = 0 \,. \tag{1.29}$$

Under condition (1.29) one obtains for an arbitrary function:

$$f(A + B) = f(A) + f'(A)B + \frac{1}{2} f''(A)[B, A] + O(B^2) \,. \tag{1.30}$$

As compared to the ordinary Taylor expansion, there are several additional terms with commutators at each order.

A use of the result above in equation (1.24) gives

$$S_i(\langle\varphi\rangle) + \frac{1}{2}S_{ijk}(\langle\varphi\rangle)\frac{\delta\langle\varphi^j\rangle}{\delta \mathrm{i}J_k} + O(\hbar^2) = -J_i \; , \tag{1.31}$$

$$S_{ij}(\langle\varphi\rangle)\frac{\delta\langle\varphi^j\rangle}{\delta J_k} = -\delta_i^k + O(\hbar) \; . \tag{1.32}$$

Here the second equation is obtained by differentiating the first one, and it tells us what is $\delta\langle\varphi\rangle/\delta J$. Up to $O(\hbar)$, it is some Green's function of the operator $S_2$. Denote this Green's function as

$$\frac{\delta\langle\varphi^j\rangle}{\delta J_k} = G^{jk} + O(\hbar) \; . \tag{1.33}$$

One can work to any order but I shall stop here. *We obtain closed equations for the mean field*:

$$S_i(\langle\varphi\rangle) + \frac{1}{2\mathrm{i}}S_{ijk}(\langle\varphi\rangle)G^{jk}(\langle\varphi\rangle) + O(\hbar^2) = -J_i \; , \tag{1.34}$$

$$S_{ij}(\langle\varphi\rangle)G^{jk}(\langle\varphi\rangle) = -\delta_i^k \; . \tag{1.35}$$

The second term in (1.34) is the loop

$$S_i(\langle\varphi\rangle) + \frac{i}{}\!\!-\!\!\bigcirc + O(\hbar^2) = -J_i \; , \tag{1.36}$$

all elements of the loop being functions of $\langle\varphi\rangle$. But two questions remain to be answered:

(i) Which Green's function is $G$?
(ii) What are the boundary conditions to the mean-field equations?

The answers are again in the Schwinger principle. Equation (1.18) tells us what are $G$ and $\langle\varphi\rangle$:

$$\frac{1}{\mathrm{i}}G^{jk} = \frac{\langle 2|\overleftarrow{T}\left(\hat{\varphi}^j\hat{\varphi}^k\right)|1\rangle}{\langle 2|1\rangle} - \langle\varphi^j\rangle\langle\varphi^k\rangle + O(\hbar) \; , \tag{1.37}$$

$$\langle\varphi^j\rangle = \frac{\langle 2|\hat{\varphi}^j|1\rangle}{\langle 2|1\rangle} \; . \tag{1.38}$$

Multiply these expressions by the coefficients that make the linear $Q$ out of $\varphi$:

$$Q(\hat{\varphi}) = k_j\hat{\varphi}^j \; , \tag{1.39}$$

and send $j$ either to $\Sigma_1$ or to $\Sigma_2$. By the definition of the states $|1\rangle$ and $|2\rangle$, one obtains

$$Q(\langle\varphi\rangle\big|_{\Sigma_1}) = q_1 \; , \qquad Q(\langle\varphi\rangle\big|_{\Sigma_2}) = q_2 \; , \tag{1.40}$$

$$k_j G^{jk}\Big|_{j\in\Sigma_1} = 0 \;, \qquad k_j G^{jk}\Big|_{j\in\Sigma_2} = 0 \;. \tag{1.41}$$

From (1.37) it follows also that

$$G^{jk} = G^{kj} \;. \tag{1.42}$$

The Green's function $G$ is symmetric and completely determined by the boundary conditions (1.41). This completes the determination of the mean-field equations (1.34), and for these equations one arrives at a boundary-value problem with the boundary conditions (1.40). As a result, the quantum boundary-value problem is reduced to a c-number boundary-value problem. I say "c-number" rather than "classical" because there are differences, and one is the presence of terms $O(\hbar)$ in the equations, but, as far as the setting of the problem is concerned, there is no difference. One arrives at the same boundary-value problem for the observable field as in the case of the classical states.

Note that the Green's function $G$ and, thereby, the mean-field equations do not depend on the eigenvalues $q$. The eigenvalues appear only in the boundary conditions to the equations. However, $G$ depends on the choice of the observables $Q$ themselves and, through them, on the choice of the states $|1\rangle$ and $|2\rangle$. Therefore, the mean-field equations are state-dependent.

Although the Green's function $G$ depends on the choice of the states, it possesses two universal properties. One has already been mentioned: $G$ is always symmetric. The other one is this. Let us make a variation in the operator $S_2$ and find out how does $G$ respond:

$$S_2 G = -1 \;,$$

$$S_2 \delta G = -\delta S_2 G \;,$$

$$\delta G =?$$

To answer this question, one can use the Schwinger principle again. The result is the following *variational law*:

$$\delta G = G \delta S_2 G \;, \tag{1.43}$$

and this law is universal. It is the same for all boundary-value problems.

The variational law (1.43) is remarkable. It is characteristic of finite-dimensional matrices. If a matrix has a unique inverse, then the inverse obeys this law. This law is valid, for example, for the inverse of an elliptic operator, i.e., for the Euclidean Green's function. It is valid also for the advanced and retarded Green's functions:

$$\delta G^+ = G^+ \delta S_2 G^+ \;, \qquad \delta G^- = G^- \delta S_2 G^- \;. \tag{1.44}$$

But it is not valid generally, and, in the case of $S_2$, it is exceptional.

The variational law for $G$ has an important implication. Namely, let us differentiate the left-hand side of the mean-field equations

$$\Gamma_i(\varphi) \equiv S_i(\varphi) + \frac{1}{2\mathrm{i}} S_{imn}(\varphi) G^{mn}(\varphi) + O(\hbar^2) \tag{1.45}$$

to see if the result is symmetric. One obtains

$$\frac{\delta \Gamma_i(\varphi)}{\delta \varphi^j} - \frac{\delta \Gamma_j(\varphi)}{\delta \varphi^i} = \frac{1}{2\mathrm{i}} S_{imn} G^{m\bar{m}} G^{n\bar{n}} S_{\bar{m}\bar{n}j} - (i \leftrightarrow j) + O(\hbar^2)$$
$$= 0 + O(\hbar^2) \ . \tag{1.46}$$

This means that $\Gamma_i(\varphi)$ is a gradient, i.e., there exists an action generating the mean-field equations:

$$\Gamma_i(\varphi) = \frac{\delta \Gamma(\varphi)}{\delta \varphi^i} \ . \tag{1.47}$$

There is another way to arrive at the same conclusion. Consider a function of the mean field defined by the Legendre transformation

$$\Gamma(\langle \varphi \rangle) = \frac{1}{\mathrm{i}} \ln \langle 2|1 \rangle - \langle \varphi^k \rangle J_k \tag{1.48}$$

where $J$ is to be expressed through $\langle \varphi \rangle$ by solving equation (1.23). It is easy to see that this function satisfies the equation

$$\frac{\delta \Gamma(\langle \varphi \rangle)}{\delta \langle \varphi^i \rangle} = -J_i \ , \tag{1.49}$$

and, therefore, its gradient is the left-hand side of the mean-field equations.

$\Gamma(\varphi)$ is the effective action. Up to $\hbar^2$ it is of the form

$$\Gamma(\varphi) = S(\varphi) + \frac{1}{2\mathrm{i}} \ln \det G(\varphi) + O(\hbar^2) \tag{1.50}$$

where the second term is the loop without external lines:

$$\Gamma(\varphi) = S(\varphi) + \bigcirc + O(\hbar^2) \ . \tag{1.51}$$

The effective action exists for any boundary-value problem but these actions are different for different such problems. Only in the classical approximation, the action and the equations are independent of the boundary conditions.

Let us go over to expectation values.

## The Quantum Initial-Value Problem

In this problem, given is only one local state (which I shall assume normalized). Since the field operators are now sandwiched between the states associated with one and the same $\Sigma$:

$$\langle 1|(\cdots)|1\rangle , \qquad \langle 1|1\rangle = 1 \tag{1.52}$$

one cannot apply the Schwinger principle: there is no room for varying the source. One can create this room artificially by inserting a complete set of states associated with some later $\Sigma$:

$$\langle 1|1\rangle = \sum_q \langle 1|2q\rangle\langle 2q|1\rangle , \tag{1.53}$$

$$\Sigma_2 > \Sigma_1$$

but this alone will not help because the source is varied in both amplitudes, and these variations cancel. It will help only if the two amplitudes in (1.53) are functions of different sources, i.e., if, instead of (1.53), one introduces a function of two independent sources, $J$ and $J^*$:

$$Z(J^*, J) = \sum_q \langle 1|2q\rangle_{J^*}\langle 2q|1\rangle_J . \tag{1.54}$$

This amounts to considering two copies of the quantum field: one with the source $J$, the other one with the source $J^*$, and using in (1.54) the amplitudes of both. Then one can vary only one source and, after that, make the sources coincident. Using the Schwinger principle, one obtains

$$\left.\frac{\delta^n Z(J^*, J)}{\delta \mathrm{i} J_{j_1}\cdots\delta \mathrm{i} J_{j_n}}\right|_{J^*=J} = \langle 1|\overleftarrow{T}\left(\hat{\varphi}^{j_1}\ldots\hat{\varphi}^{j_n}\right)|1\rangle . \tag{1.55}$$

In this way the expectation values can be calculated.

The technique of two sources is called time-loop formalism because in expression (1.54) one goes forward in time, from $\Sigma_1$ to some $\Sigma_2$, and then back from $\Sigma_2$ to $\Sigma_1$ but with another copy of the quantum field.

For every partial amplitude in (1.54) we have equation (1.20)

$$\left(S_i\left(\frac{\delta}{\delta \mathrm{i} J}\right) + J_i\right)\langle 2q|1\rangle_J = 0 . \tag{1.56}$$

Since the other amplitude in (1.54) does not depend on $J$, we can linearly combine equations (1.56) to obtain

$$\left(S_i\left(\frac{\delta}{\delta \mathrm{i} J}\right) + J_i\right)Z(J^*, J) = 0 . \tag{1.57}$$

Only one source is active in this differential equation. The other one is a parameter. Therefore, we can just repeat the consideration above with $Z(J^*, J)$ in place of $\langle 2|1\rangle$, and in this way derive the mean-field equations. We obtain the loop expansion of exactly the same form as before:

$$S_i(\langle\varphi\rangle) + \frac{1}{2\mathrm{i}}S_{ijk}(\langle\varphi\rangle)G^{jk}(\langle\varphi\rangle) + O(\hbar^2) = -J_i , \tag{1.58}$$

$$S_{ij}(\langle\varphi\rangle)G^{jk}(\langle\varphi\rangle) = -\delta_i^k \; , \tag{1.59}$$

and in these loops we must make the sources coincident. There are only two elements in all loops, $\langle\varphi\rangle$ and $G$. Upon setting $J^* = J$, $\langle\varphi\rangle$ becomes the genuine expectation value

$$\langle\varphi^k\rangle = \left.\frac{\delta\ln Z(J^*, J)}{\delta\mathrm{i}J_k}\right|_{J^*=J} = \langle 1|\hat\varphi^k|1\rangle \; , \tag{1.60}$$

and the matrix $G$ is given by the expression

$$\frac{1}{\mathrm{i}}G^{jk} + O(\hbar) = \left.\frac{\delta^2\ln Z(J^*, J)}{\delta\mathrm{i}J_j\delta\mathrm{i}J_k}\right|_{J^*=J} = \langle 1|\overleftarrow{T}\left(\hat\varphi^j\hat\varphi^k\right)|1\rangle - \langle\varphi^j\rangle\langle\varphi^k\rangle \; . \tag{1.61}$$

I am using for it the same letter $G$ but it is now a different Green's function of the operator $S_2$. Equations (1.58) with this Green's function in all loops are the expectation-value equations.

The solution of the expectation-value equations is specified completely by the initial conditions on $\Sigma_1$ following from (1.60) but it is not easy to write these conditions down in the general terms. Only half of them is obvious: the $Q$'s on $\Sigma_1$ are given. To obtain the other half, one would need to find the variables canonically conjugate to $Q$'s and calculate their expectation values on $\Sigma_1$.[2] The same concerns the specification of the Green's function $G$. This issue will be considered in the next lecture where a different approach to it will be used.

Let us consider the state-independent properties of $G$. First, as seen from (1.61), $G$ is symmetric for any initial-value problem:

$$G^{jk} = G^{kj} \; . \tag{1.62}$$

Second, one can apply the Schwinger principle to derive the variational law for $G$. At this point, the initial-value problem differs significantly from the boundary-value problem. When the operator $S_2$ is varied in the generating function (1.54), one can no longer play with only one source because $S_2$ is the same for both copies of the quantum field, and, therefore, both amplitudes in (1.54) respond. As a consequence, all four matrices of second derivatives are generally involved:

$$\frac{\delta^2\ln Z}{\delta\mathrm{i}J_j\delta\mathrm{i}J_k} \; , \quad \frac{\delta^2\ln Z}{\delta\mathrm{i}J_j^*\delta\mathrm{i}J_k^*} \; , \quad \frac{\delta^2\ln Z}{\delta\mathrm{i}J_j^*\delta\mathrm{i}J_k} \; , \quad \frac{\delta^2\ln Z}{\delta\mathrm{i}J_j\delta\mathrm{i}J_k^*} \; , \tag{1.63}$$

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[2] Let $Q$'s be Hermitian, and let $P$'s have c-number commutators with $Q$'s: $[P, Q] = \mathrm{i}$. Then the expectation values in the state (1.13) satisfy the initial conditions

$$\left.\langle Q\right|_{\Sigma}\rangle = \int dq\,\overline{\Psi}(q)q\Psi(q) \; , \qquad \left.\langle P\right|_{\Sigma}\rangle = \mathrm{i}\int dq\,\overline{\Psi}(q)\frac{\partial}{\partial q}\Psi(q)$$

where the overline means complex conjugation. If both $Q(\hat\varphi)$ and $P(\hat\varphi)$ are linear, these are initial conditions directly for $\langle\varphi\rangle$.

i.e., the Green's function $G^{jk}$, its complex conjugate, and two Wightman functions: $\langle 1|\hat{\varphi}^j\hat{\varphi}^k|1\rangle$ and its transpose. The Wightman functions can be expressed through $G^{jk}$ and the advanced or retarded Green's function:

$$\mathrm{i}\langle 1|\hat{\varphi}^j\hat{\varphi}^k|1\rangle - \mathrm{i}\langle\varphi^j\rangle\langle\varphi^k\rangle = G^{jk} - G^{+jk} + O(\hbar) = G^{kj} - G^{-kj} + O(\hbar) \ . \quad (1.64)$$

The result of the calculation is the following variational law for $G$:

$$\delta G = G^-\delta S_2 G + G\delta S_2 G^+ - G^-\delta S_2 G^+ \ . \quad (1.65)$$

It is no more the simple law (1.43) but it is, nevertheless, universal because $G^+$ and $G^-$ are state-independent. The variational law (1.65) is valid for any initial-value problem.

The left-hand side of the expectation-value equations has the form (1.45) as before but, since the variational law for $G$ is different, the former inference about the symmetry of $\delta\Gamma_i/\delta\varphi^j$ needs to be revised. This inference is no longer valid. The advanced and retarded Green's functions arrange it so that

$$\frac{\delta\Gamma_i(\varphi)}{\delta\varphi^j} = 0 \quad \text{when } i < j \quad (1.66)$$

and

$$\frac{\delta\Gamma_i(\varphi)}{\delta\varphi^j} \neq 0 \quad \text{when } i > j \ . \quad (1.67)$$

It follows that there is no action generating the expectation-value equations.

The nonexistence of an action for the initial-value problem is seen also from the consideration of the Legendre transform of the generating function (1.54). It is now a function of two fields:

$$\Gamma(\varphi^*, \varphi) = \frac{1}{\mathrm{i}}\ln Z(J^*, J) - \varphi J + \varphi^* J^* \quad (1.68)$$

where

$$\varphi = \frac{\delta\ln Z(J^*, J)}{\delta\mathrm{i}J} \ , \qquad \varphi^* = -\frac{\delta\ln Z(J^*, J)}{\delta\mathrm{i}J^*} \ . \quad (1.69)$$

The expectation-value equations are obtained as

$$\varphi = \langle 1|\hat{\varphi}|1\rangle \ : \qquad \left.\frac{\delta\Gamma(\varphi^*, \varphi)}{\delta\varphi^i}\right|_{\varphi^*=\varphi} = -J_i \ , \quad (1.70)$$

and, therefore,

$$\Gamma_i(\varphi) = \left.\frac{\delta\Gamma(\varphi^*, \varphi)}{\delta\varphi^i}\right|_{\varphi^*=\varphi} \ . \quad (1.71)$$

This is *not* a gradient.

## 2 The In-Vacuum State and Schwinger–Keldysh diagrams

### Specification of The State

In order to proceed, I need to specify the state. This will be done in several steps.

*Step 1.* It will be assumed that $S_2$ is a second-order hyperbolic operator, and the energy-momentum tensor of the field of small disturbances $\delta\varphi^i$ with the action

$$\frac{1}{2}S_{ij}\delta\varphi^i\delta\varphi^j \tag{2.1}$$

satisfies the dominant energy condition.

*Step 2.* The initial-value surface will be shifted to the remote past:

$$\Sigma_1 \to -\infty \ . \tag{2.2}$$

Consider the operator field equations (1.2)–(1.3):

$$J_i + S_i(c) + S_{ij}(c)(\hat{\varphi}-c)^j + \sum_{n=2}^{\infty}\frac{1}{n!}S_{ij_1\cdots j_n}(c)(\hat{\varphi}-c)^{j_1}\ldots(\hat{\varphi}-c)^{j_n} = 0 \ . \tag{2.3}$$

If $c^i$ is some classical solution:

$$S_i(c) = -J_i \ , \tag{2.4}$$

and $\hat{\phi}^i$ is an operator solution of $S_2$ against the background $c^i$:

$$S_{ij}(c)\hat{\phi}^j = 0 \ , \tag{2.5}$$

then the field

$$\hat{\varphi}^i = c^i + \hat{\phi}^i \ , \qquad i \in \Sigma \to -\infty \tag{2.6}$$

solves the operator dynamical equations asymptotically in the remote past. It is a property of $S_2$ that its solution with smooth data having a compact support or decreasing at the spatial infinity decreases also in the timelike directions. Then, as $i \in \Sigma \to -\infty$, the nonlinear terms in (2.3) decrease even faster and are negligible. Thus, to build a Hilbert space of states, it suffices to build a representation of the algebra of $\hat{\phi}$'s.

*Step 3.* A Fock space will be built associated with the linear field $\hat{\phi}^i$. This amounts to expanding $\hat{\phi}^i$ in some basis of solutions of $S_2(c)$:

$$S_2(c)\chi_A = 0 \ , \tag{2.7}$$

$$\hat{\phi}^i = \chi_A^i \hat{a}_{\text{in}}{}^A + \overline{\chi}_A^i \hat{a}_{\text{in}}^{+\,A} \tag{2.8}$$

where the overline means complex conjugation, and the basis functions $\chi_A^i$ are normalized with the aid of the inner product:

$$(\chi_A, \chi_B) = 0 , \qquad (\overline{\chi}_A, \chi_B) = \delta_{AB} , \tag{2.9}$$

$$(\phi_1, \phi_2) \equiv -\mathrm{i} \int_\Sigma \phi_1 W_\mu \phi_2 \, d\Sigma^\mu . \tag{2.10}$$

Here $W_\mu$ is the Wronskian of $S_2$. In this way, the concept is introduced of *some* particles detectable in the past. What kind of particles are these, i.e., what kind of detectors detect these particles – depends on the choice of the basis of solutions but, in any case, the following functions will be chosen for the local observables $Q$:

$$Q^A(\hat{\varphi}\big|_\Sigma) = -\mathrm{i}\delta^{AB} \int_\Sigma \chi_B W_\mu (\hat{\varphi} - c) \, d\Sigma^\mu , \tag{2.11}$$

$$\Sigma \to -\infty .$$

One needs these observables only on the initial-value surface, and, there, they coincide with the annihilation operators of the introduced particles:

$$Q^A(\hat{\varphi}\big|_{\Sigma \to -\infty}) = \hat{a}_{\mathrm{in}}{}^A . \tag{2.12}$$

The choice of the quantum state will be made in favour of the zero-eigenvalue eigenstate of these observables:

$$\hat{a}_{\mathrm{in}}{}^A|1\rangle = 0 . \tag{2.13}$$

This is the vacuum of the introduced particles.

It follows from (2.6) and (2.8) that the field's expectation value in the state (2.13), when taken in the remote past, coincides with the classical solution $c^i$:

$$\langle 1|\hat{\varphi}^i|1\rangle = c^i , \qquad i \in \Sigma \to -\infty . \tag{2.14}$$

The ad hoc classical solution $c^i$ can then be eliminated completely both from the asymptotic form of the quantum field

$$\hat{\varphi}^i = \langle \varphi^i \rangle + \hat{\phi}^i , \qquad i \in \Sigma \to -\infty \tag{2.15}$$

and from the equation defining the Fock modes

$$S_{ij}(\langle \varphi \rangle)\hat{\phi}^j = 0 , \qquad i \in \Sigma \to -\infty . \tag{2.16}$$

Only the mean field itself figures as a background.

The specification of the state is, however, not completed because the mean field in the past remains an arbitrary classical solution:

$$S_i(\langle \varphi \rangle) = -J_i , \qquad i \in \Sigma \to -\infty \tag{2.17}$$

and the state itself remains the vacuum of undefined particles. To make the final determination, one more step is needed.

*Step 4.* The final choice of the state assumes one more limitation on the original action. Namely, it will be assumed that the external source $J_i$ and all the external fields that may be present in the action $S$ are asymptotically static in the past. This means that, asymptotically in the past, there exists a vector field $\xi^\mu$ such that it is nowhere tangent to any of the Cauchy surfaces, and the Lie derivative in the direction of $\xi^\mu$ of all external fields is zero. Specifically,

$$\mathcal{L}_\xi J_i = 0 \,, \qquad i \in \Sigma \to -\infty \,. \tag{2.18}$$

If this limitation is fulfilled, then, among the solutions of (2.17) for the mean field in the past, there is the static one:

$$\mathcal{L}_\xi \langle \varphi^i \rangle = 0 \,, \qquad i \in \Sigma \to -\infty \,. \tag{2.19}$$

Choose it. Next, use the fact that, with this choice, the operator $S_2(\langle \varphi \rangle)$ commutes with the Lie derivative, and choose for the basis solutions of $S_2(\langle \varphi \rangle)$ the functions that, asymptotically in the past, are eigenfunctions of the Lie derivative:

$$\mathrm{i}\mathcal{L}_\xi \chi_A^i = \varepsilon_A \chi_A^i \,, \quad \varepsilon_A > 0 \,, \quad i \in \Sigma \to -\infty \,. \tag{2.20}$$

This fixes both the initial conditions for the mean field and the type of particles whose vacuum is the chosen state. These are particles with definite energies.

Since $S_2$ is a second-order hyperbolic operator, it contains some tensor field, $g^{\mu\nu}$, contracting the second derivatives. The inverse matrix, $g_{\mu\nu}$, can serve and does serve in every respect as a metric on the base manifold. The metric enters the original action $S$ either as a part of the quantum field $\hat\varphi^i$ or as an external field. In both cases it is subject to equation (2.19). When applied to the metric, this is the Killing equation. Thus, we assume the existence, asymptotically in the past, of a timelike Killing vector $\xi^\mu$.

The specification of the quantum initial data is now completed. The notation for the state defined above is

$$|1\rangle = |\text{in vac}\rangle \,, \tag{2.21}$$

and its full name is relative standard in-vacuum state. It is "relative" because it is relative to the background generated by an asymptotically static source. It is "standard" because it refers to the standard concept of particles. It is "in" because these particles are incoming. And it is "vacuum" because these particles are absent.

The state should not necessarily be chosen as the zero-eigenvalue eigenstate. Since the expectation-value equations do not depend on the eigenvalues, they will have the same form for any eigenstate of the annihilation operators, i.e., for any coherent state

$$\hat a_{\text{in}}{}^A |\text{in } \alpha\rangle = \alpha^A |\text{in } \alpha\rangle \,. \tag{2.22}$$

Only the initial conditions for the mean field will be different:

$$\langle \alpha \text{ in}|\hat{\varphi}^i|\text{in } \alpha\rangle = c^i + \chi_A^i \alpha^A + \overline{\chi}_A^i \overline{\alpha}^A \, , \qquad i \in \Sigma \to -\infty \, . \tag{2.23}$$

In addition to the static background $c^i$ generated by a source, the mean field in the past contains now the incoming wave of an arbitrary profile. This is the general setting of the classical evolution problem for an observable field like the electromagnetic or gravitational field. The fact that the nature of the state has changed from classical to quantum did not affect this setting.

It will be useful to keep comparing the initial-value problem with the boundary-value problem. In the latter case, one can define similarly the out-vacuum state and specify the quantum boundary data as

$$|1\rangle = |\text{in vac}\rangle \, , \qquad |2\rangle = |\text{out vac}\rangle \, . \tag{2.24}$$

**Perturbation Theory**

With this specification of the states, let us come back to the mean-field equations. There remains to be obtained the Green's function $G(\varphi)$ that figures in the loops. We need it for an arbitrary background $\varphi$ but we have a variational law, (1.43) or (1.65), which may be regarded as a differential equation for $G(\varphi)$ with respect to $\varphi$. The only thing that is missing and that depends on the choice of states is the initial condition to this equation. It suffices, therefore, to know $G$ for only one background.

Then let us do the simplest: perturbation theory around the trivial background. A second-order hyperbolic operator with the trivial background is the D'Alembert operator with flat metric, $\square_0$:

$$S_2(\varphi) = \square_0 + P \, . \tag{2.25}$$

The remainder is a perturbation $P$.

In the case of the boundary-value problem, the variational law is (1.43), and, therefore, the expansion of $G(\varphi)$ is of the form

$$G(\varphi) = G_0 + G_0 P G_0 + G_0 P G_0 P G_0 + \ldots \tag{2.26}$$

where $G_0$ is $G$ for the trivial background. This expansion is to be inserted in the loop in the mean-field equations

$$\frac{1}{2\mathrm{i}} S_{ijk}(\varphi) G^{jk}(\varphi) = \frac{i}{}\!-\!\bigcirc \, . \tag{2.27}$$

Let for simplicity $P$ be a potential. One obtains the loop expanded in powers of $P$:

$$\frac{x}{}\!-\!\bigcirc = \int dy_1 \ldots dy_n \, F(x|y_1, \ldots y_n) P(y_1) \ldots P(y_n) \, . \tag{2.28}$$

The coefficients $F$ will be called formfactors. The formfactors are loop diagrams

$$F(x|y) = \quad {}^{x}\!\!\bigcirc\!\!{}^{y} \quad , \tag{2.29}$$

$$F(x|y_1, y_2) = \quad {}^{x}\!\!\triangleleft\!\!\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} \quad , \tag{2.30}$$

with the same propagator for all lines: the trivial-background Green's function

$$\overline{\qquad} = G_0 . \tag{2.31}$$

What is $G_0$? With the trivial background and the standard in- and out- vacuum states, it is the Feynman Green's function:

$$G_0 = G_{\text{FEYNMAN}} . \tag{2.32}$$

Let us do the same thing for the initial-value problem. The loop in the expectation-value equations will, in the same way, be expanded in powers of the perturbation, and the expansion will have the same form (2.28), but the formfactors will be different because the variational law for $G$ is different. It is now (1.65) rather than (1.43). Using this law, one obtains for the formfactors three diagrams in place of one:

$$F(x|y) = \quad {}^{x}\!\!\bigcirc\!\!{}^{y} \; + \; {}^{x}\!\!\bigcirc\!\!{}^{y} \; - \; {}^{x}\!\!\bigcirc\!\!{}^{y} \quad , \tag{2.33}$$

five diagrams in place of one:

$$F(x|y_1, y_2) = \quad {}^{x}\!\!\triangleleft\!\!\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} \; + \; {}^{x}\!\!\triangleleft\!\!\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} \; + \; {}^{x}\!\!\triangleleft\!\!\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}$$

$$- \; {}^{x}\!\!\triangleleft\!\!\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} \; - \; {}^{x}\!\!\triangleleft\!\!\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} \quad , \tag{2.34}$$

and so on. There are two types of propagators in these diagrams: the trivial-background $G$, and the trivial-background retarded or advanced Green's function. Respectively, there are two types of lines:

$$\overline{\qquad} = G_0 , \qquad \overleftarrow{\qquad} = G_0^- \text{ or } G_0^+ . \tag{2.35}$$

In the latter case, the arrow points the direction of growth of time. And what is now $G_0$? In terms of the linear field (2.5) it is

$$\frac{1}{\mathrm{i}}G_0^{jk} = \langle \mathrm{in\ vac}|\overleftarrow{T}(\hat{\phi}^j\hat{\phi}^k)|\mathrm{in\ vac}\rangle\Big|_{\text{trivial background}} \tag{2.36}$$

and differs from the previous case in that the "$\langle \mathrm{out\ vac}|$" is replaced by the "$\langle \mathrm{in\ vac}|$". But, with the trivial background, the vacuum for the linear field is stable. The out-vacuum coincides with the in-vacuum. Therefore,

$$G_0 = G_{\text{FEYNMAN}} \quad (\text{again!})\ . \tag{2.37}$$

The diagrams above are called Schwinger–Keldysh diagrams. There is not more than one Feynman propagator in every diagram. The remaining ones are the retarded and advanced Green's functions organized in a special way and with special signs of the diagrams themselves. There is a mystery in this special arrangement. What do these diagrams want to tell us? We must disclose their secret because working with them directly is not what can be recommended.

### Mystery of The Schwinger–Keldysh Diagrams

One thing is obvious right away. In the diagrams above, there is always a chain of retarded Green's functions connecting a given point $y$ with the observation point $x$. Therefore, the formfactor vanishes if at least one of the $y$'s is in the future of $x$. This is the *retardation property*

$$F(x|y_1,\ldots y_n) = 0 \quad \text{when}\ \ y_m > x\ ,\ \ \forall m\ . \tag{2.38}$$

But this is true of every Schwinger–Keldysh diagram, and why do they appear in the special combinations? What is the role of the Feynman propagator?

Let us make a Fourier transformation of the formfactor with respect to the differences $(x - y_m)$ in the Minkowski coordinates:

$$F(x|y_1,\ldots y_n) = \int dk_1\ldots dk_n\ \exp\Big(\mathrm{i}\sum_{m=1}^{n} k_m(x - y_m)\Big) f(k_1,\ldots k_n)\ . \tag{2.39}$$

How come that $F$ possesses the retardation property? It is only that $f$ should admit an analytic continuation to the upper half-plane in the timelike components of $k$'s. Then, for $y_m$ later than $x$, we shall be able to close the integration contour in the upper half-plane of $k_m^0$, and the integral will vanish. There should be a function of complex momenta $f(z_1,\ldots z_n)$ analytic in the upper half-planes of $z_m^0$ and such that $f(k_1,\ldots k_n)$ is its limiting value on the real axes:

$$f(k_1,\ldots k_n) = f(z_1,\ldots z_n)\Big|_{z_m^0\ =\ k_m^0\ +\ \mathrm{i}\varepsilon}\ . \tag{2.40}$$

Let us build this function.

All diagrams in a given-order formfactor are similar. They all are integrals over the momentum circulating in the loop, and the integrands are identical.

The difference is only in the integration contours. Thus any diagram in the lowest-order formfactor $f(k)$ is of the form

$$\text{(diagram)}^k = \int d\boldsymbol{p} \int_{\mathcal{C}} dp^0 \, \frac{\text{polynomial in momenta}}{\left(-p^{02} + \boldsymbol{p}^2\right)\left(-(p^0 - k^0)^2 + (\boldsymbol{p} - \boldsymbol{k})^2\right)} \; . \quad (2.41)$$

There are, generally, as many factors in the denominator as there are propagators in the loop, and each factor contains two poles. The contour $\mathcal{C}$ passes round them in accordance with the type of the propagator. One of the three rules applies to each pair of poles:

 retardation rule,

 advancement rule,

 Feynman rule.

Let us now shift the external momentum $k^0$ to the complex plane. The poles will shift to the complex plane but we shall also deform smoothly the contour so that it do not cross the poles. In this way one can build a function of complex momenta for each Schwinger–Keldysh diagram. Thus the lowest-order formfactor with complex momentum, $f(z)$, is a sum of three functions:

$$f(z) = \int d\boldsymbol{p} \int_{\mathcal{C}_1} dp^0 \, (\ldots) + \int d\boldsymbol{p} \int_{\mathcal{C}_2} dp^0 \, (\ldots) - \int d\boldsymbol{p} \int_{\mathcal{C}_3} dp^0 \, (\ldots) , \quad (2.42)$$

and the contours $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ for $z^0$ in the upper half-plane are shown in Fig. 1. By considering the pinch conditions, i.e., the conditions that the poles pinch the integration contour, one can check in each case that these functions can have singularities only on the real axis. Therefore, if we consider them in the upper half-plane, they are analytic, and their limits on the real axis are our original diagrams.

There remains to be understood what are these functions. Since the integrands are identical, the sum of the integrals in (2.42) is the integral over the sum of the contours

$$f(z) = \int d\boldsymbol{p} \int_{\mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_3} dp^0 \, (\ldots) . \quad (2.43)$$

Sum up the three contours in Fig. 1. The resultant contour is such that every pair of poles is passed round by the Feynman rule. It may be called Feynman contour.

But the Feynman contour defines also the in-out formfactor (2.29) in which both propagators are Feynman, except that the in-out formfactor is not the
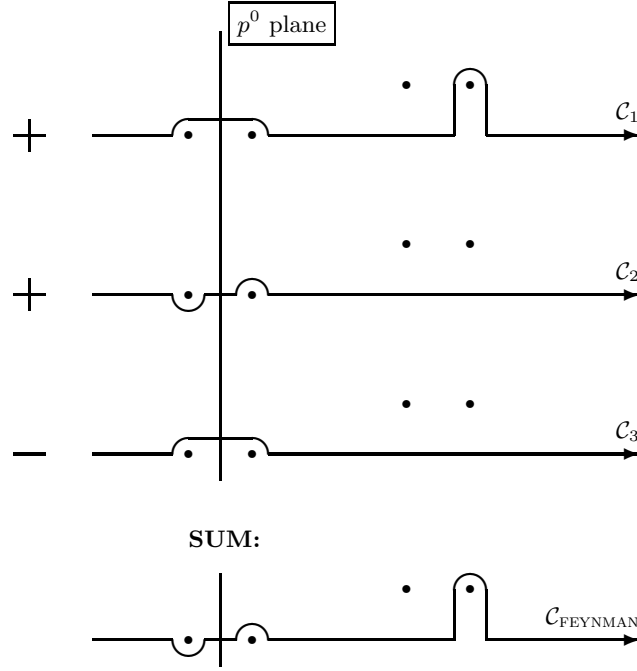
**Fig. 1.** Integration contours for the three diagrams in the lowest-order formfactor (2.42). The sum of the contours is the Feynman contour.

limit of $f(z)$ from the upper half-plane. It is this limit on only half of the real axis, and on the other half it is the limit from the lower half-plane. The in-in and in-out formfactors are different boundary values of the same complex function having a cut on the real axis:

$$\text{in-in} \ : \qquad f(k) = f(z)\Big|_{z^0 \, = \, k^0 \, + \, \mathrm{i}\varepsilon} \, , \tag{2.44}$$

$$\text{in-out} \ : \qquad f(k) = f(z)\Big|_{z^0 \, = \, (1 \, + \, \mathrm{i}\varepsilon)k^0} \, , \tag{2.45}$$

and the function itself is the integral over the Feynman contour

$$f(z) = \int d\boldsymbol{p} \int_{\mathcal{C}_{\text{FEYNMAN}}} dp^0 \, (\ldots) \, . \tag{2.46}$$

The same is true of all $n$-th order formfactors, and this is a disclosure of the mystery. In each case, the set of Schwinger–Keldysh diagrams is just a splitting of one Feynman diagram whose purpose is to display the retardation property and in this way to tell us which boundary value is to be taken.

### Reduction to The Euclidean Effective Action

The Feynman contour is famous for the fact that, when the external momenta are on the imaginary axis, the Feynman contour is the imaginary axis itself. With all the momenta imaginary, both the external ones and the one circulating in the loop, this is the Euclidean formfactor. Then we can *start* with the calculation of the Euclidean formfactor and next analytically continue it in momenta from the imaginary axis to the real axis either in the way shown in Fig. 2(a) or in the way shown in Fig. 2(b). In the first case we shall obtain the in-out formfactor, and in the second case the in-in formfactor of Lorentzian theory. It is invaluable that loops can be calculated Euclidean.



(a) **IN-OUT**                    (b) **IN-IN**
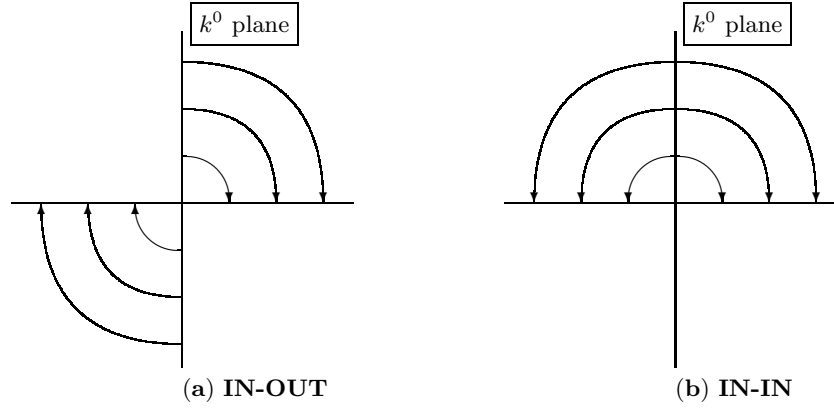
**Fig. 2.** Analytic continuation of the Euclidean formfactor that gives (**a**) the in-out formfactor, (**b**) the in-in formfactor of Lorentzian theory.

Then let us make one more step. A formfactor with the Euclidean momentum can be put in the spectral form

$$f(k) = \int\limits_0^\infty dm^2 \, \frac{\rho(m^2)}{m^2 + k^2} \; + \; \text{a polynomial in } k^2 \;, \qquad (2.47)$$

$$k^2 > 0$$

with some spectral weight $\rho(m^2)$, the resolvent $1/(m^2 + k^2)$, and a polynomial accounting for a possible growth of $f(k)$ at $k^2 \to \infty$. There are similar forms for the higher-order formfactors. If the formfactor is in the spectral form, the procedure of analytic continuation boils down merely to replacing the Euclidean resolvent with the retarded or Feynman resolvent:

$$\text{in-in}: \qquad f(k) = \int\limits_0^\infty dm^2 \, \frac{\rho(m^2)}{m^2 - (k^0 + i\varepsilon)^2 + \boldsymbol{k}^2} \; + \; \text{a polynomial in } k^2 \;,$$

$$(2.48)$$

$$\text{in-out}\ :\qquad f(k) = \int\limits_0^\infty dm^2\, \frac{\rho(m^2)}{m^2 - k^{02} + \boldsymbol{k}^2 - \mathrm{i}\varepsilon} + \ \text{a polynomial in } k^2\ .$$

$$(2.49)$$

Note that the spectral weight is the same in all cases: the one of the Euclidean loop. Thus, the problem boils down to obtaining the spectral weights of the Euclidean formfactors.

Then back from the Fourier-transformed formfactors to the formfactors themselves, and from the formfactors to the mean-field equations. For the loop in these equations expanded in powers of the perturbation, we obtain an expression of the following form:

$$
\begin{aligned}
{}^{x}\!\!-\!\!\bigcirc\ &= (c_1 + c_2 \Box_0 + \ldots)P(x) \\
&+ \int\limits_0^\infty dm^2\, \rho(m^2)\frac{1}{m^2 - \Box_0}P(x) \\
&+ \int\limits_0^\infty dm_1^2 dm_2^2 dm_3^2\, \rho(m_1^2, m_2^2, m_3^2) \\
&\quad \times \frac{1}{m_1^2 - \Box_0}\left[\left(\frac{1}{m_2^2 - \Box_0}P(x)\right)\left(\frac{1}{m_3^2 - \Box_0}P(x)\right)\right] \\
&+ \ldots\ .
\end{aligned}
$$

$$(2.50)$$

Here the first term is local. It comes from the polynomial in the spectral form. The remaining terms are nonlocal but expressed through the resolvent which is a Green's function of the massive operator $\Box_0 - m^2$. It is initially the Euclidean Green's function since we are calculating the Euclidean loop. For the Lorentzian equations, we arrive at the following rule. To obtain the expectation-value equations in the in-vacuum state, replace all the Euclidean resolvents in (2.50) with the retarded Green's functions. To obtain the mean-field equations for the in-out problem, replace all the Euclidean resolvents with the Feynman Green's functions:

$$
\text{All } \frac{1}{m^2 - \Box_0}\ 
\begin{array}{l}
\nearrow\ \text{Euclidean,} \\
\longrightarrow\ \text{Retarded,} \\
\searrow\ \text{Feynman.}
\end{array}
\qquad (2.51)
$$

At every level of expectation-value theory, there are proofs that the expectation-value equations possess two basic properties: they are real and causal. Causality is the retardation property discussed above. But it is not enough to have proofs. These properties should be manifestly built into the working formalism. Expression (2.50) offers such a formalism. Since the retarded resolvent secures the causality and is real, this expression is manifestly real and causal.

But even this is not enough. The theory may possess symmetries, and one may want these symmetries to be manifest. To this end it will be noted that, although expansion (2.50) is obtained in terms of the trivial-background resolvent $1/(m^2 - \Box_0)$, it can be regrouped so as to restore the full-background resolvent

$$\frac{1}{m^2 - S_2} = \frac{1}{m^2 - \Box_0 - P} \tag{2.52}$$

at each order. It does not matter whether this regrouping will be made in the expectation-value equations or in the Euclidean equations because the retarded and Euclidean Green's functions obey the same variational law (1.43):

$$\frac{1}{m^2 - \Box_0} = \frac{1}{m^2 - S_2} - \frac{1}{m^2 - S_2} P \frac{1}{m^2 - S_2} + \dots \, . \tag{2.53}$$

This proves that the rule of replacing resolvents applies to the full-background resolvents as well as to the trivial-background ones. The latter fact is important because the Euclidean loops can be calculated covariantly from the outset, and the transition to the expectation-value equations by replacing the full-background resolvents does not break the manifest symmetries. The expectation-value equations are obtained in as good an approximation as the Euclidean equations are.

There remains to be made a final observation. For the Euclidean equations, *there is* an effective action:

$$\overset{i}{-\!\bigcirc} = \frac{\delta}{\delta \varphi^i} \bigcirc \tag{2.54}$$

because the variational law for the Euclidean Green's function is (1.43). It is invaluable that loops can be calculated without external lines. This reduces the calculations greatly, helps to control symmetries, helps to control renormalizations.

Thus, at the end of the day, we conclude that *there is* an action that generates the expectation-value equations but it does so indirectly, i.e., *not* through the least-action principle. To make this clear, consider (for the illustrative purposes only) any quadratic action:

$$\Gamma(\varphi) = \frac{1}{2} \int dx \, \varphi f(\Box_0) \varphi \, .$$

Whatever the operator $f(\Box_0)$ is, in the variational derivative it gets symmetrized:

$$\frac{\delta \Gamma(\varphi)}{\delta \varphi} = \frac{1}{2} \left( f(\Box_0) + f^{\mathrm{T}}(\Box_0) \right) \varphi = f^{\mathrm{sym}}(\Box_0) \varphi \, .$$

Assuming that the function $f(\Box_0)$ is in the spectral form

$$f(\Box_0) = \int\limits_0^\infty dm^2 \, \rho(m^2) \frac{1}{m^2 - \Box_0} \, ,$$

one obtains the variational equations with the symmetrized resolvent:

$$\int\limits_0^\infty dm^2\, \rho(m^2)\left(\frac{1}{m^2-\Box_0}\right)^{\mathrm{sym}}\varphi = -J\ .$$

These cannot be the expectation-value equations since they are not causal. But, through the derivation above, we know how to correct this: just to replace the symmetrized resolvent with the retarded resolvent. The corrected equations

$$\int\limits_0^\infty dm^2\, \rho(m^2)\left(\frac{1}{m^2-\Box_0}\right)^{\mathrm{ret}}\varphi = -J\ .$$

do not already follow from any action although indirectly they do. Only if the action $\Gamma(\varphi)$ is local, i.e., the function $f(\Box_0)$ is polynomial, the least-action principle holds directly.

Two precepts should be kept in mind when using the formalism above. First, the replacement rule concerns the resolvents of the formfactors and not the propagators in the loop. The loop should be calculated Euclidean. Hence

*First Precept:* first do the loop, next replace the resolvents.

Second, the replacement of resolvents is to be made in the equations and not in the action. It does not make sense to make it in the action. Hence

*Second Precept:* first vary the action, next replace the resolvents.

We thus go over to the calculation of the Euclidean effective action.

## 3 The Effective Action

### The Operator $S_2$

The $\varphi^i$ is a set of fields for which a more explicit notation will now be used:

$$\varphi^i = \varphi^a(x)\ . \tag{3.1}$$

The operator $S_2$ acts on a small disturbance of $\varphi^i$ and is a second-order differential operator

$$S_{ij}\delta\varphi^j = \left(X_{ab}^{\mu\nu}\partial_\mu\partial_\nu + Y_{ab}^\mu\partial_\mu + Z_{ab}\right)\delta\varphi^b(x)\ . \tag{3.2}$$

The generality of this operator will, however, be restricted by the condition that the coefficient of the senior term factorizes as

$$X_{ab}^{\mu\nu} = \omega_{ab}\, g^{\mu\nu}\ , \qquad \det\omega_{ab} \neq 0\ ,\ \ \det g^{\mu\nu}\neq 0\ . \tag{3.3}$$

In this case, the operator (3.2) is said to be diagonal, or minimal, or nonexotic. Condition (3.3) is too restrictive and not necessary. It can be replaced by a more general condition

$$\det\left(X_{ab}^{\mu\nu}n_\mu n_\nu\right) = C(g^{\mu\nu}n_\mu n_\nu)^d \quad \forall n_\mu \ , \quad d = \dim a \ , \ \ C \neq 0 \ , \ \ \det g^{\mu\nu} \neq 0 \ , \tag{3.4}$$

and even this condition can be generalized. Higher-order and first-order operators can also be considered but, in all of these cases, the Green's functions of $S_2$ are expressed through the Green's functions of a diagonal second-order operator. The case (3.3) is basic.

In the case (3.3), the matrix $\omega_{ab}$ can be factored out:

$$S_{ij}\delta\varphi^j = \omega_{ac}H_b^c\delta\varphi^b(x) \ , \tag{3.5}$$

and a covariant derivative can be introduced:

$$\nabla_\mu\,\delta\varphi^a = (\delta_b^a\partial_\mu + \mathcal{A}_{\mu\,b}^{\ a})\,\delta\varphi^b \tag{3.6}$$

so as to absorb the first-order term:

$$H_b^a = \delta_b^a g^{\mu\nu}\nabla_\mu\nabla_\nu + P_b^a \ . \tag{3.7}$$

This is the final form of $S_2$. A short notation will be used:

$$H = \square\hat{1} + \hat{P} \tag{3.8}$$

where

$$\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu \ , \tag{3.9}$$

and the hat designates a matrix in $a, b$:

$$\hat{1} = \delta_b^a \ , \quad \hat{P} = P_b^a \ , \quad \operatorname{tr}\hat{P} = P_a^a \ , \quad \text{etc.} \tag{3.10}$$

The matrix $\omega_{ab}$ may be regarded as a local metric in the space of fields. The symmetry of $S_2$ implies that this matrix is symmetric, covariantly constant, and converts $\hat{P}$ into a symmetric form:

$$\omega_{ab} = \omega_{ba} \ , \qquad \nabla_\mu\omega_{ab} = 0 \ , \tag{3.11}$$

$$P_a^c\omega_{cb} - P_b^c\omega_{ca} = 0 \ . \tag{3.12}$$

The dominant energy condition implies that $\omega_{ab}$ is positive definite. The matrix $g^{\mu\nu}$ is the inverse of the metric on the base manifold. Since we are considering Euclidean theory, this metric is positive definite too.

Apart from the algebraic factor $\omega_{ac}$ in (3.5), the operator $S_2$ contains three background fields:

$$g^{\mu\nu} \ , \quad \nabla_\mu \ , \quad \hat{P} \tag{3.13}$$

i.e., the metric, the connection (or covariant derivative), and the matrix potential. And where is the original background $\varphi$ of $S_2(\varphi)$? When $S_2$ is calculated

from the action $S$, the metric, connection, and potential are obtained as functions of the original set of fields $\varphi$, but from now on it does not matter. The effective action is expressed in a universal manner through the fields (3.13) only.

The strengths of the fields (3.13) are respectively the Riemann tensor, the commutator of covariant derivatives, and the potential which is its own strength:

$$R_{\alpha\beta\mu\nu} \, , \quad [\nabla_\mu, \nabla_\nu] = \hat{\mathcal{R}}_{\mu\nu} \, , \quad \hat{P} \, . \tag{3.14}$$

I shall call these field strengths curvatures and use for them the collective notation

$$\left( R_{\alpha\beta\mu\nu} \, , \ \hat{\mathcal{R}}_{\mu\nu} \, , \ \hat{P} \right) = \Re \, . \tag{3.15}$$

The following contractions of the curvatures will be called currents:

$$\hat{J}_\mu \equiv \nabla^\nu \hat{\mathcal{R}}_{\mu\nu} \, , \tag{3.16}$$

$$J_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \, , \quad J \equiv g^{\mu\nu} J_{\mu\nu} \, . \tag{3.17}$$

The currents are conserved:

$$\nabla^\mu \hat{J}_\mu = 0 \, , \qquad \nabla^\mu J_{\mu\nu} = 0 \, . \tag{3.18}$$

If all the curvatures vanish, the background is trivial. The effective action is a functional of the curvatures (3.15).

### Redundancy of The Curvatures

The effective action is a nonlocal functional of the curvatures, and this fact conditions a certain simplification.

Since the commutator curvature is a commutator, it satisfies the Jacobi identity, and so does the Riemann curvature:

$$\nabla_\gamma \hat{\mathcal{R}}_{\mu\nu} + \nabla_\nu \hat{\mathcal{R}}_{\gamma\mu} + \nabla_\mu \hat{\mathcal{R}}_{\nu\gamma} = 0 \, , \tag{3.19}$$

$$\nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\gamma\mu} + \nabla_\mu R_{\alpha\beta\nu\gamma} = 0 \, . \tag{3.20}$$

Act on these identities with $\nabla^\gamma$. In the first term, the operator $\Box$ forms, and in the remaining terms commute the covariant derivatives. The commutator brings an extra power of the curvature. The equations obtained

$$\Box \hat{\mathcal{R}}_{\mu\nu} + O(\Re^2) = 2\nabla_{[\nu} \hat{J}_{\mu]} \, , \tag{3.21}$$

$$\Box R_{\alpha\beta\mu\nu} + O(\Re^2) = 4\nabla_{[\mu} \nabla_{\langle\alpha} \left( J_{\nu]\beta\rangle} - \frac{1}{2} g_{\nu]\beta\rangle} J \right) \tag{3.22}$$

hold identically and have the form of inhomogeneous wave equations, the role of inhomogeneity being played by the currents. In (3.21), (3.22), the brackets of both types $[\,]$ and $\langle\,\rangle$ denote the antisymmetrization in the respective indices.

The equations (3.21) and (3.22) are nonlinear but they can be solved by iteration. The result is that the commutator and Riemann curvatures get expressed in a nonlocal fashion through their currents and an arbitrary solution of the homogeneous wave equation

$$\Box \hat{\mathcal{R}}_{\mu\nu}^{\text{wave}} = 0 \ , \qquad \Box R_{\alpha\beta\mu\nu}^{\text{wave}} = 0 \ . \tag{3.23}$$

If the metric is Lorentzian, this solution is fixed by initial data which can be given in the remote past. It follows that the commutator and Riemann curvatures are specified by giving an incoming wave and the current $J$. This fact underlies the Maxwell and Einstein equations. They fix the currents $J$. Adding initial conditions to these equations specifies the connection and metric.

In the present case, since the metric is Euclidean, there are no wave solutions:

$$\hat{\mathcal{R}}_{\mu\nu}^{\text{wave}} = 0 \ , \qquad R_{\alpha\beta\mu\nu}^{\text{wave}} = 0 \ , \tag{3.24}$$

and the Green's function $1/\Box$ is unique. Therefore, the commutator and Riemann curvatures are expressed entirely through their currents:

$$\hat{\mathcal{R}}_{\mu\nu} = \frac{1}{\Box} 2\nabla_{[\nu}\hat{J}_{\mu]} + O(J^2) \ , \tag{3.25}$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{\Box} 4\nabla_{[\mu}\nabla_{\langle\alpha} \left( J_{\nu]\beta\rangle} - \frac{1}{2}g_{\nu]\beta\rangle}J \right) + O(J^2) \ . \tag{3.26}$$

Thus, the curvatures are redundant because there are no waves in Euclidean theory. Owing to this fact, the set of field strengths (3.15) reduces to

$$\left( J_{\mu\nu} \ , \ \hat{J}_\mu \ , \ \hat{P} \right) \ , \tag{3.27}$$

and the effective action is a functional of the reduced set.

## The Axiomatic Effective Action

To what class of functionals does the effective action belong? One can say in advance that this should be a functional analytic in the curvature. Indeed, the first variational derivative of the effective action taken at the trivial background should vanish because, in the absence of an external source, the relative vacuum becomes the absolute vacuum. The trivial background should solve the mean-field equations in the absolute vacuum. Higher-order variational derivatives taken at the trivial background determine the correlation functions in the absolute vacuum. They may not vanish but neither should they blow up.

The analyticity suggests that the effective action can be built as a sum of nonlocal invariants of $N$-th order in the curvature:

$$\Gamma = \sum_N \Gamma_N \ , \qquad \Gamma_N = O[\Re^N] \ . \tag{3.28}$$

Nonlocal invariant is, however, an uncertain concept. Even local invariant of $N$-th order in the curvature is a concept that needs to be refined but this is easy to do. The most general local monomial that can be built out of the available quantities yields an invariant of the form

$$\int dx\, g^{1/2} \underbrace{(\nabla_1 \dots \nabla_1)(\nabla_2 \dots \nabla_2) \dots}_{k} \Re_1 \Re_2 \dots \Re_N + O[\Re^{N+1}]\,. \qquad (3.29)$$

This monomial is a product of $N$ curvatures and $k$ covariant derivatives, all indices being contracted by the metric. In (3.29), the labels $1, 2, \dots$ point out which derivative acts on which curvature but all the curvatures are at the same point, and the total number of derivatives is finite. Of course, the curvature sits also in the covariant derivatives and in the metric that contracts the indices. Therefore, the $N$-th order invariant can only be defined up to terms $O[\Re^{N+1}]$. In particular, the covariant derivatives in (3.29) can be commuted freely because the contribution of a commutator is already $O[\Re^{N+1}]$.

One may now consider a class of nonlocal invariants that can formally be represented as infinite series of local invariants:

$$\Gamma_N = \int dx\, g^{1/2} \sum_{k=0}^{\infty} c_k \underbrace{(\nabla_1 \dots \nabla_1)(\nabla_2 \dots \nabla_2) \dots}_{k} \Re_1 \Re_2 \dots \Re_N + O[\Re^{N+1}]\,. \quad (3.30)$$

Here $c_k$ are some dimensional constants. It can be seen that this is the needed class[3]. The number of curvatures in (3.30) is $N$ but the number of derivatives is unlimited. Only a finite number of derivatives can contract with the curvatures. The remaining ones can only contract among themselves. If two derivatives acting on the same curvature contract, they make a $\square$ operator acting on this curvature:

$$\nabla_1{}^2 = \square_1\,, \quad \nabla_2{}^2 = \square_2\,, \quad \dots\,. \qquad (3.31)$$

If two derivatives acting on different curvatures contract, the contraction can again be written in terms of the $\square$ operators:

$$\begin{aligned} 2\nabla_1\nabla_2 &= (\nabla_1 + \nabla_2)^2 - \nabla_1{}^2 - \nabla_2{}^2 \\ &= \square_{1+2} - \square_1 - \square_2 \end{aligned} \qquad (3.32)$$

but there appears a $\square$ operator acting on the product of two curvatures:

$$\square_{1+2}\, \Re_1 \Re_2 \Re_3 \dots = \square\,(\Re\Re)\, \Re_3 \dots\,. \qquad (3.33)$$

As a result, (3.30) takes the form

---

[3] To see it, consider any diagram with massive propagators and expand it formally in the inverse mass. The method that accomplishes this expansion is known as the Schwinger–DeWitt technique.

$$\Gamma_N = \int dx\, g^{1/2} \left( \sum_{k_1,k_2,\cdots=0}^{\infty} c_k (\Box_1)^{k_1} (\Box_2)^{k_2} (\Box_{1+2})^{k_3} \cdots \right)$$

$$\times \underbrace{\left( \nabla...\Re_1 \nabla...\Re_2 \ldots \nabla...\Re_N \right)}_{\text{contraction}} + O[\Re^{N+1}]\,.$$

$$(3.34)$$

There remains an infinite series in the $\Box$ variables, and these variables themselves are operators acting on the curvatures in a given contraction. The remaining series is some function of the $\Box$ variables:

$$\Gamma_N = \int dx\, g^{1/2} F\left( \Box_1, \Box_2, \Box_{1+2}, \ldots \right) \underbrace{\left( \nabla...\Re_1 \nabla...\Re_2 \ldots \nabla...\Re_N \right)}_{\text{contraction}} + O[\Re^{N+1}]\,.$$

$$(3.35)$$

This is the general form of a nonlocal invariant of $N$-th order in the curvature. The function $F$ is a formfactor.

There is, in addition, the identity

$$\nabla_1 + \nabla_2 + \ldots + \nabla_N = 0 \qquad (3.36)$$

which reduces the number of variables in the function $F$. The sum in (3.36) is a derivative acting on the product of all curvatures, i.e., a total derivative. Total derivatives vanish because the curvatures may be considered having compact supports. Thus invariants of first order in the curvature can only be local because any derivative is a total derivative. Therefore, the first-order formfactors are constants:

$$N = 1\ : \qquad F = \text{const.} \qquad (3.37)$$

At the second order, all formfactors are functions of only one argument because the remaining arguments can be eliminated by integration by parts:

$$N = 2\ : \qquad F = F(\Box_1)\,, \qquad (3.38)$$

$$\Box_2 = \Box_1\,, \qquad \Box_{1+2} = 0\,.$$

At the third order, all formfactors are functions of three individual $\Box$'s because the $\Box$'s acting on pairs can be eliminated:

$$N = 3\ : \qquad F = F(\Box_1, \Box_2, \Box_3)\,, \qquad (3.39)$$

$$\Box_{1+2} = \Box_3\,, \qquad \Box_{1+3} = \Box_2\,, \qquad \Box_{2+3} = \Box_1\,.$$

The $\Box$'s acting on pairs appear beginning with the fourth order in the curvature and are parameters of the on-shell scattering amplitudes.

Nonlocal invariants of a given order make a linear space in which all possible contractions of $N$ curvatures and their derivatives make a basis, and the formfactors play the role of coefficients of the linear combining. The basis can be built by listing all independent contractions. The effective action is an expansion in this basis with certain coefficients–formfactors:

$$\Gamma = \Gamma_{\mathrm{I}} + \Gamma_{\mathrm{II}} + \Gamma_{\mathrm{III}} + \dots \,, \tag{3.40}$$

$$\Gamma_{\mathrm{I}} = \int dx\, g^{1/2} \Big[ c_1 R + c_2 \operatorname{tr} \hat{P} \Big] \,, \tag{3.41}$$

$$\begin{aligned}
\Gamma_{\mathrm{II}} = \int dx\, g^{1/2} \operatorname{tr} \Big[\, & R_{\mu\nu}\, F_1(\Box)\, R^{\mu\nu} \\
& + R\, F_2(\Box)\, R \\
& + \hat{P}\, F_3(\Box)\, R \\
& + \hat{P}\, F_4(\Box)\, \hat{P} \\
& + \hat{\mathcal{R}}_{\mu\nu}\, F_5(\Box)\, \hat{\mathcal{R}}^{\mu\nu} \Big] \,,
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
\Gamma_{\mathrm{III}} = \int dx\, g^{1/2} \operatorname{tr} \Big[\, & F_1(\Box_1,\Box_2,\Box_3)\, \hat{P}_1 \hat{P}_2 \hat{P}_3 \\
& + F_2(\Box_1,\Box_2,\Box_3)\, \hat{\mathcal{R}}_1{}^{\mu}{}_{\alpha} \hat{\mathcal{R}}_2{}^{\alpha}{}_{\beta} \hat{\mathcal{R}}_3{}^{\beta}{}_{\mu} \\
& + \cdots \\
& + F_{29}(\Box_1,\Box_2,\Box_3)\, \nabla_\lambda \nabla_\sigma R_1^{\alpha\beta} \nabla_\alpha \nabla_\beta R_2^{\mu\nu} \nabla_\mu \nabla_\nu R_3^{\lambda\sigma} \Big] \,.
\end{aligned} \tag{3.43}$$

In the first-order action (3.41), there are 2 basis contractions: the Ricci scalar and the trace of the matrix potential, and the formfactors are constants. In the second-order action, there are 5 independent contractions listed in (3.42). In the third-order action, there are 29 basis contractions, examples of which are given in (3.43). Here I shall stop because, for the problems of interest, the third order is sufficient. The reason for that will be explained in the next lecture.

In the expressions above, the basis invariants are written in terms of the curvatures but they can be rewritten in terms of the conserved currents. Note also that the operator arguments of the third-order formfactors $F$ commute because they act on different objects. Since the arguments commute, the functions $F$ themselves are ordinary functions of three variables.

Thus, even before any calculation, we have an ansatz for the effective action, with unknown formfactors. We need them in the spectral forms

$$F_{\mathrm{k}}(\Box) = \int_0^\infty dm^2\, \frac{\rho_{\mathrm{k}}(m^2)}{m^2 - \Box} + \text{ a polynomial in } \Box \,, \tag{3.44}$$

$$F_{\mathrm{k}}(\square_1, \square_2, \square_3) = \int\limits_0^\infty dm_1^2 dm_2^2 dm_3^2 \; \frac{\rho_{\mathrm{k}}(m_1^2, m_2^2, m_3^2)}{(m_1^2 - \square_1)(m_2^2 - \square_2)(m_3^2 - \square_3)} \; , \quad (3.45)$$

and then we can proceed directly to the expectation-value equations. Unknown are only the spectral weights. These are to be calculated from the loop diagrams but there is an alternative approach. One can look for the general limitations on the spectral weights stemming from axiomatic theory. These limitations may be sufficient to solve one's expectation-value problem. In this case, the solution will prove to be independent of the details of the quantum-field model and the approximations made in it. Moreover, the effective action above does not refer even to quantum field theory. It is an action for the observable field, and its implications may be valid irrespective of the underlying fundamental theory. Only certain axiomatic properties of the spectral weights may be important. There is an example in which this approach has been implemented [53].

Here, the axiomatic approach will not be considered. Let us see how the effective action is calculated from loops.

**Heat Kernel**

Consider any diagram in the effective action



$$, \quad (3.46)$$

and, for every propagator, write

$$\text{———} = -\frac{1}{H} = \int\limits_0^\infty ds \, \mathrm{e}^{sH} \; . \quad (3.47)$$

The kernel of the exponential operator

$$\mathrm{e}^{sH} \delta(x, y) \equiv \hat{K}(x, y|s) \quad (3.48)$$

(and the operator itself) is called heat kernel, and the parameter $s$ is often called proper time. Both names are matters of history, and a matter of physics is the fact that $H$ is negative definite. The matrix $P$ in (3.8) may spoil the negativity but, since it is treated perturbatively, as one of the curvatures, this does not matter.

Upon the insertion of (3.47), the diagram remains the same as before but with the heat kernels in place of the propagators, and the integrations over the proper times will be left for the last:

$$\left( \text{⬲} \right) = \int\limits_0^\infty ds_1 \ldots \int\limits_0^\infty ds_n \left( \text{⬲}_{s_1 \cdots s_n} \right) . \qquad (3.49)$$

The one-loop effective action is the functional trace of the heat kernel, integrated over $s$:

$$\bigcirc = \frac{1}{2} \ln \det \frac{1}{H} = \frac{1}{2} \int_0^\infty \frac{ds}{s} \int dx \ \mathrm{tr} \ \hat{K}(x, x|s) . \qquad (3.50)$$

Thus, one is left with diagrams with the heat kernels. It will be seen in a moment why this is better.

The expansion rule for the exponential operator has already been considered in (1.27). There remains to be presented the lowest-order approximation for the heat kernel:

$$\hat{K}(x, y|s) = \frac{1}{(4\pi s)^{D/2}} \left( e^{-\sigma(x,y)/2s} \hat{a}(x, y) + O[\Re] \right) , \qquad (3.51)$$

$$D = \text{dimension of the base manifold.} \qquad (3.52)$$

At the lowest order in the curvature, the potential $P$ does not affect this expression but the metric and connection do. As mentioned above, covariant expansions cannot be rigid. In (3.51):

$$2\sigma(x, y) = (\text{geodetic distance between } x \text{ and } y)^2 \qquad (3.53)$$

in the metric entering the operator $H$. The connection entering the operator $H$ defines a parallel transport along a line. Parallel transport is a linear mapping, so there exists a propagator of parallel transport (the matrix that accomplishes this mapping). In (3.51):

$$\hat{a}(x, y) = \text{propagator of the parallel transport from } y \text{ to } x \qquad (3.54)$$
$$\text{along the geodesic connecting } y \text{ and } x.$$

The geodesic comes from the metric, and the parallel transport from the connection.

The two-point functions (3.53) and (3.54) are the main elements of the Schwinger–DeWitt technique mentioned above and the basic building blocks for all Green's functions: of the hyperbolic operator $H$, and of the elliptic operator $H$, and the heat kernel. What is special about the heat kernel? Special is the fact that, as seen from expression (3.51), the heat kernel is finite at the coincident points. Green's functions of the hyperbolic and elliptic operators are singular, and this is normal. Abnormal is the fact that in the loop diagrams they appear at the coincident points. Finiteness of the heat kernel at the coincident points is a bonus owing to which all diagrams with the heat kernels are finite.

The divergences of the loop diagrams reappear in the proper-time integrals in (3.49). These integrals diverge at the lower limits. At this stage, one more advantage of the heat kernel comes into effect. Namely, the manifold dimension $D$ enters only the overall factor in (3.51). Apart from this factor, the expansion of the heat kernel in the curvature does not contain $D$ explicitly. Therefore, loops with the heat kernels are calculated once for all dimensions, and then the knowledge of the analytic dependence on $D$ enables one to apply the dimensional regularization to the proper-time integrals. One integrates by parts in $s$ keeping $\mathrm{Re}\, D < 4$ and next goes over to the limit $D \to 4$. For example,

$$\int\limits_0^\infty \frac{ds}{s^{D/2-1}} f(s) = \frac{1}{2-D/2} f(0) - \int\limits_0^\infty ds\, \ln s \frac{df(s)}{ds} + O\left(2 - D/2\right) . \quad (3.55)$$

The dimensional regularization annihilates all power divergences. Only the logarithmic divergences survive and take the form of poles in dimension. These poles affect only the polynomial terms in the spectral representations of the formfactors. They appear in the coefficients of the polynomials, thereby making these coefficients indefinite. As a consequence, the local terms of the effective action will have indefinite coefficients. I shall come back to this issue.

After the substitution of the heat kernels for the propagators, the calculation of loops becomes an entertaining geometrical exercise.

## Loops and Geometry

The heat kernel involves $\sigma$ and $\hat{a}$. The derivative of $\sigma$

$$\nabla^\mu \sigma(x,y) \equiv \sigma^\mu(x,y) \qquad \qquad \sigma^\mu(x,y) \qquad (3.56)$$

is the vector tangent to the geodesic connecting $y$ and $x$, directed outwards, and normalized to the geodetic distance between $y$ and $x$:

$$g_{\mu\nu}\sigma^\mu\sigma^\nu = 2\sigma , \qquad \sigma^\mu\Big|_{x=y} = 0 , \qquad \det \nabla^\nu \sigma^\mu \Big|_{x=y} \neq 0 . \quad (3.57)$$

The normalization condition is a closed equation for $\sigma$ which together with the conditions at the coincident points can serve as the definition of $\sigma$. The defining equation for $\hat{a}$ together with the condition at the coincident points is

$$\sigma^\mu \nabla_\mu \hat{a}(x,y) = 0 , \qquad \hat{a}\Big|_{x=y} = \hat{1} . \quad (3.58)$$

The determinant

$$\det\left(\nabla_\mu^x \nabla_\nu^y \sigma(x,y)\right) = g^{1/2}(x) g^{1/2}(y) \Delta(x,y) \quad (3.59)$$

is known as the Van Vleck–Morette determinant. It is responsible, in particular, for a caustic of the geodesics emanating from $x$ or $y$.

The vector $\sigma^\mu$ can be used to expand any function in a covariant Taylor series. For a scalar, this series is of the form

$$f(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sigma^{\mu_1} \ldots \sigma^{\mu_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} f(x) . \tag{3.60}$$

If $f$ is not a scalar, it should at first be parallel transported from $y$ to $x$:

$$f(y) = \hat{a}(y,x) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sigma^{\mu_1} \ldots \sigma^{\mu_n} \nabla_{\mu_1} \ldots \nabla_{\mu_n} f(x) . \tag{3.61}$$

The covariant Taylor expansion is a regrouping of the ordinary Taylor expansion. Whatever the connection is, it cancels in this series. The series can formally be written in the exponential form

$$f(y) = \hat{a}(y,x) \exp\left(-\sigma^\mu \nabla_\mu\right) f(x) \tag{3.62}$$

which will be of use below. Two-point functions expanded in this way get expressed through their covariant derivatives at the coincident points. Thus

$$\Delta(x,y) = 1 + \frac{1}{6} R_{\mu\nu} \sigma^\mu \sigma^\nu + \ldots . \tag{3.63}$$

A loop always involves the ring of $\hat{a}$'s

$$\hat{a}(x,x_1)\hat{a}(x_1,x_2)\ldots\hat{a}(x_n,x) , \qquad \langle\!\!\langle \, \rangle\!\!\rangle \tag{3.64}$$

i.e., the parallel transport around a geodetic polygon. The ring of two $\hat{a}$'s is the parallel transport there and back along the same path. Therefore,

$$\hat{a}(x,x_1)\hat{a}(x_1,x) \equiv \hat{1} . \tag{3.65}$$

The ring of three $\hat{a}$'s is the parallel transport around the geodetic triangle. It involves the commutator curvature, and the curvature terms can be calculated:

$$\hat{a}(x,x_1)\hat{a}(x_1,x_2)\hat{a}(x_2,x) = \hat{1} + \frac{1}{2}\hat{\mathcal{R}}_{\alpha\beta}\sigma_1{}^\alpha \sigma_2{}^\beta + \ldots , \tag{3.66}$$



$$. \tag{3.67}$$

This is sufficient because any polygon can be broken into triangles:

$$\tag{3.68}$$

Solution of the geodetic triangle is also involved. In the notation of (3.67),
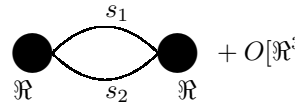
$$\left(\sigma^\mu(x_1, x_2)\right)^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2\sigma_1\sigma_2 - \frac{1}{3}R_{\mu\alpha\nu\beta}\sigma_1{}^\mu\sigma_1{}^\nu\sigma_2{}^\alpha\sigma_2{}^\beta + \dots . \tag{3.69}$$

Here the first two terms make the Pythagorean theorem, the third term accounts for the angle not being the right angle, and the terms with the Riemann curvature can be calculated.

The above is to give a flavour of what loops imply.

### Calculation of Loops

The heat kernel calculates loops with a remarkable elegance. As an example, consider the contribution of the second order in the curvature to the effective action. The respective one-loop diagram contains two curvatures $\Re$ and two heat kernels with the proper times $s_1$ and $s_2$:



$$= \int dx\, g^{1/2} \int dy\, g^{1/2}\, \Re(x)\hat{K}(x, y|s_1)\hat{K}(x, y|s_2)\Re(y) + O[\Re^3] . \tag{3.70}$$

Suppose that the calculation only needs to be done with accuracy $O[\Re^3]$. Then one can insert in (3.70) the lowest-order approximation for the heat kernels. In this approximation, the rings of $\hat{a}$'s collapse to $\hat{1}$, and the remaining $\hat{a}$'s always transport the $\Re$'s to the same point arranging their complete contraction. With the $\hat{a}$'s and the numerical coefficients omitted, the diagram (3.70) is of the form

$$\frac{1}{s_1{}^{D/2}}\frac{1}{s_2{}^{D/2}} \int dx\, g^{1/2} \int dy\, g^{1/2}$$

$$\times\, \Re(x)\exp\left(-\frac{\sigma(x, y)}{2s_1}\right)\exp\left(-\frac{\sigma(x, y)}{2s_2}\right)\Re(y) . \tag{3.71}$$

But the exponents here simply add, and the two heat kernels turn into one with a complicated proper-time argument:

$$\frac{1}{(s_1 s_2)^{D/2}} \int dx\, g^{1/2} \int dy\, g^{1/2}\, \Re(x)\exp\left(-\frac{s_1 + s_2}{2s_1 s_2}\sigma(x, y)\right)\Re(y)$$

$$= \frac{1}{(s_1 + s_2)^{D/2}} \int dx\, g^{1/2} \int dy\, g^{1/2}\, \Re(x)K\left(x, y\bigg|\frac{s_1 s_2}{s_1 + s_2}\right)\Re(y) . \tag{3.72}$$

One only needs to rewrite this heat kernel in the operator form:

$$\frac{1}{(s_1 + s_2)^{D/2}} \int dx\, g^{1/2}\, \Re \exp\left(\frac{s_1 s_2}{s_1 + s_2}\Box\right) \Re(y)\,, \qquad (3.73)$$

and the loop is done. The proper-time integral

$$\int\limits_0^\infty ds_1 \int\limits_0^\infty ds_2 \frac{1}{(s_1 + s_2)^{D/2}} \exp\left(\frac{s_1 s_2}{s_1 + s_2}\Box\right) = F(\Box) \qquad (3.74)$$
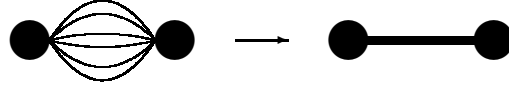
is the formfactor.

What has happened? The propagators in the loop glued together, and the loop turned into a tree:

     (3.75)

This is what means to do the loop. *It means to turn it into a tree.* The role of the propagator in the tree is played by the formfactor $F(\Box)$.

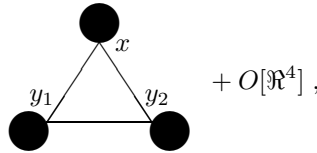Consider now any multi-loop diagram with parallel propagators. It turns into a tree

     (3.76)

in a completely similar way. The inverse proper times add:

$$\frac{1}{s_1} + \frac{1}{s_2} + \ldots = \frac{1}{s_{\text{total}}}$$

(the law of parallel conductors). There is nothing to do.

For more than two curvatures a more powerful method is used. Consider the diagram

     $+\, O[\Re^4]\,,$     (3.77)

and suppose again that it is needed only up to the next order in the curvature. Then, with the $\hat{a}$'s and the numerical coefficients omitted, it is of the form

$$\frac{1}{s_1{}^{D/2}} \frac{1}{s_2{}^{D/2}} \frac{1}{s_3{}^{D/2}} \int dx\, g^{1/2} \int dy_1\, g^{1/2} \int dy_2\, g^{1/2}$$

$$\times \exp\left(-\frac{\sigma(x,y_1)}{2s_1} - \frac{\sigma(x,y_2)}{2s_2} - \frac{\sigma(y_1,y_2)}{2s_3}\right) \Re(x)\Re(y_1)\Re(y_2)\,. \quad (3.78)$$

Choose one of the vertices, say $x$, to be the observation point of the effective Lagrangian. One of the curvatures, $\Re(x)$, is already there. Shift the remaining curvatures to $x$ using the covariant Taylor series:

$$\Re(y_i) = \exp\left(-\sigma_i{}^\mu \nabla_\mu\right)\Re(x)\,, \tag{3.79}$$

$$\sigma_i{}^\mu = \sigma^\mu(x,y_i)\,,\quad i=1,2\,. \tag{3.80}$$

Next, consider the geodetic triangle with the same vertices as in the diagram. For the geodesics connecting $x$ with $y_i$, write

$$2\sigma(x,y_i) = (\sigma_i)^2\,, \tag{3.81}$$

and, for the geodesic between the $y$'s, use the Pythagorean theorem:

$$2\sigma(y_1,y_2) = (\sigma_1)^2 + (\sigma_2)^2 - 2\sigma_1\sigma_2 + O[\Re]\,. \tag{3.82}$$

Finally, replace the integration variables:

$$y_1{}^\mu \to \sigma_1{}^\mu\,,\quad y_2{}^\mu \to \sigma_2{}^\mu\,. \tag{3.83}$$

The Jacobian

$$\left|\frac{\partial\sigma^\mu(x,y_i)}{\partial y_i{}^\nu}\right|^{-1} = \frac{g^{1/2}(x)}{g^{1/2}(y_i)}\Delta^{-1}(x,y_i) = \frac{g^{1/2}(x)}{g^{1/2}(y_i)}\left(1+O[\Re]\right) \tag{3.84}$$

removes the measure $g^{1/2}$ from the integral in $y_i$ and brings an extra $g^{1/2}$ to the integral in $x$. Expression (3.78) takes the form

$$\frac{1}{(s_1s_2s_3)^{D/2}}\int dx\,g^{1/2}\left(g^{1/2}(x)\right)^2\int d\sigma_1 d\sigma_2 \exp\left(-\frac{\sigma_1{}^2}{4s_1} - \frac{\sigma_2{}^2}{4s_2}\right.$$
$$\left.-\frac{\sigma_1{}^2 + \sigma_2{}^2 - 2\sigma_1\sigma_2}{4s_3} - \sigma_1{}^\mu\nabla_\mu{}^1 - \sigma_2{}^\mu\nabla_\mu{}^2\right)\Re(x)\Re_1(x)\Re_2(x)\,. \tag{3.85}$$

Here the labels $1,2$ on $\nabla_\mu$ and $\Re$ point out which $\nabla_\mu$ acts on which $\Re$. The operators $\nabla_\mu$ figure as parameters in the integral, and, up to the next order in $\Re$, they commute. Since the parameters commute, the integral in $\sigma_1{}^\mu$, $\sigma_2{}^\mu$ is an ordinary Gaussian integral. Do it. The extra factor $\left(g^{1/2}(x)\right)^2$ cancels, and the result is

$$B(s_1,s_2,s_3)\int dx\,g^{1/2}\exp\left(\sum_{i,k=1}^2 b_{ik}(s_1,s_2,s_3)\nabla_i\nabla_k\right)\Re(x)\Re_1(x)\Re_2(x)$$
$$\tag{3.86}$$

where $B(s_1,s_2,s_3)$ is some function of the proper times, and the exponent is a quadratic form in $\nabla_1$, $\nabla_2$ with $s$-dependent coefficients. The loop is done. The integral

$$\int\limits_{0}^{\infty} ds_1 ds_2 ds_3 \, B(s_1, s_2, s_3) \exp\left(\sum_{i,k=1}^{2} b_{ik}(s_1, s_2, s_3) \nabla_i \nabla_k\right)$$

$$= F(\nabla_1{}^2, \nabla_2{}^2, \nabla_1\nabla_2) \quad (3.87)$$

is the formfactor. Integration by parts in $x$ brings it to the $\Box$ arguments:

$$F(\nabla_1{}^2, \nabla_2{}^2, \nabla_1\nabla_2) \to F(\nabla_1{}^2, \nabla_2{}^2, \nabla^2) \, . \quad (3.88)$$

The effect of the calculation above is again that the loop is turned into a tree:



$$(3.89)$$
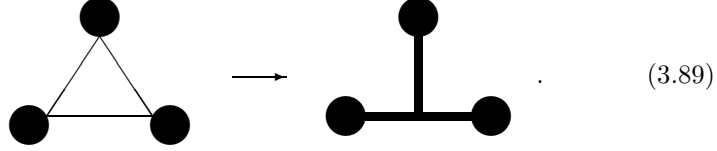
The vertex of the tree is the formfactor $F(\nabla_1{}^2, \nabla_2{}^2, \nabla_3{}^2)$. This method applies to any diagram with the heat kernels. One only needs to do Gaussian integrals, and the result is always the exponential of a quadratic combination of $\nabla$'s. The formfactor is a function of the products $\nabla_i \nabla_k$.

### The One-Loop Formfactors

The result of the proper-time integrations depends essentially on the dimension $D$. For $D = 4$, the one-loop formfactors in the effective action (3.40) are as follows.

With one exception, all second-order formfactors are logs:

$$F_1(\Box) = \frac{1}{60} \frac{1}{2(4\pi)^2} \ln(-\Box) + \text{ const. }, \quad (3.90)$$

$$F_2(\Box) = -\frac{1}{180} \frac{1}{2(4\pi)^2} \ln(-\Box) + \text{ const. }, \quad (3.91)$$

$$F_3(\Box) = \frac{1}{18} \frac{1}{2(4\pi)^2} \, , \quad (3.92)$$

$$F_4(\Box) = \frac{1}{2} \frac{1}{2(4\pi)^2} \ln(-\Box) + \text{ const. }, \quad (3.93)$$

$$F_5(\Box) = \frac{1}{12} \frac{1}{2(4\pi)^2} \ln(-\Box) + \text{ const.} \quad (3.94)$$

Since

$$-\ln(-\Box) = \int\limits_{0}^{\infty} dm^2 \, \frac{1}{m^2 - \Box} + \text{ const. }, \quad (3.95)$$

these expressions have the spectral forms (3.44) with definite spectral weights and indefinite additive constants (polynomials of the zeroth power). Respectively, the effective action contains a set of local terms with unspecified coefficients:

$$\Gamma = \frac{1}{2(4\pi)^2} \int dx\, g^{1/2} \Big( c_1 R + c_2 \operatorname{tr} \hat{P} + c_3 R_{\mu\nu} R^{\mu\nu} + c_4 R^2$$
$$+ c_5 \operatorname{tr}(\hat{P}\hat{P}) + c_6 \operatorname{tr}(\hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}^{\mu\nu}) + \frac{1}{18} R \operatorname{tr} \hat{P} + \text{ nonlocal terms} \Big) . \quad (3.96)$$

The nonlocal terms are specified completely.

The third-order formfactors have no polynomial terms and indefinite coefficients. The simplest third-order formfactor is $F_1(\square_1, \square_2, \square_3)$ in (3.43). It has the spectral form (3.45), and its spectral weight $\rho_1(m_1^2, m_2^2, m_3^2)$ is obtained as follows. Consider a triangle of three spectral masses



$A = $ area of the triangle.
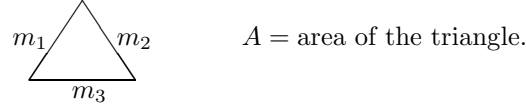
It can be built only if every mass is smaller than the sum of the two others. The spectral weight $\rho_1$ is zero if the triangle cannot be built. Otherwise, it is proportional to the inverse area of this triangle:

$$\rho_1(m_1^2, m_2^2, m_3^2) = -\frac{1}{3} \frac{1}{2(4\pi)^2} \frac{1}{4\pi A}$$
$$\times \theta(m_1 + m_2 - m_3)\theta(m_1 + m_3 - m_2)\theta(m_2 + m_3 - m_1) . \quad (3.97)$$

The remaining 28 third-order formfactors are expressed through $F_1$ and are tabulated [36]. The tables contain various integral representations of the formfactors, and their asymptotics.

The loop of the minimal second-order operator with arbitrary metric, connection, and potential is called standard loop because every calculation with it is done once, and the results can be tabulated. A calculation in any specific model boils down to combining the standard loops and using the tables. A number of recipes for the reduction to minimal operators can be found in [24]. Doing loops becomes a business similar to doing integrals.

The fact that some coefficients in the effective action remain unspecified is none of the tragedy. The effective action is a phenomenological object intended for obtaining the values of observables. The spectral weights are certain phenomenological characteristics of the vacuum like the permittivity of a medium. They are to be calculated from a more fundamental microscopic theory. Some microscopic theory of some level is incapable of specifying some of the coefficients. So what? Classical theory was capable of even less, and, nevertheless, celestial mechanics has been successfully worked up[4]. The only important question is whether the lack of knowledge affects the problems that we want to solve. This will be cleared up in the next lecture.

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[4] Remarkably, without a knowledge of string theory!

# 4 Vacuum Currents and The Effect of Particle Creation

**Vacuum Currents**

Consider quantum electrodynamics. In this case, $\varphi^a(x)$ is a set of the vector connection field and the electron–positron field

$$\text{QED:} \quad \varphi^a = \left(\mathcal{A}_\mu, \psi\right) . \tag{4.1}$$

The commutator curvature is, up to a coefficient, the Maxwell tensor, and the operator field equations are of the form

$$\nabla^\nu \mathcal{R}_{\nu\mu}(\hat{\mathcal{A}}) + J_\mu(\hat{\psi}) = -J_\mu^{\text{ext}} \tag{4.2}$$

where $J_\mu(\hat{\psi})$ is the operator electron–positron current, and $J_\mu^{\text{ext}}$ is an external source. Averaging these equations over the in-vacuum state, one obtains, according to the general derivation above, the same terms but as functions of the mean field plus a set of loops:

$$\nabla^\nu \mathcal{R}_{\nu\mu}(\langle\mathcal{A}\rangle) + J_\mu(\langle\psi\rangle) + \overset{\mathcal{A}}{\underset{\mathcal{A}}{\bigcirc}} + \overset{\psi}{\underset{\mathcal{A}}{\bigcirc}} + \overset{\mathcal{A}}{\underset{\psi}{\bigcirc}} + \overset{\psi}{\underset{\psi}{\bigcirc}} = -J_\mu^{\text{ext}} . \tag{4.3}$$

There is another such equation, for $\psi$, but, since $\psi$ has no external source, its solution is

$$\langle\psi\rangle = 0 . \tag{4.4}$$

Then, in (4.3), $J_\mu(\langle\psi\rangle)$ vanishes, and the loops with the vertices $S_{\mathcal{A}\mathcal{A}\psi}$ vanish. There are no such vertices in QED but, if there were, as in gravidynamics, they would be proportional to $\langle\psi\rangle$ and vanish by (4.4). The photon loop also vanishes because neither there is a vertex $S_{\mathcal{A}\mathcal{A}\mathcal{A}}$ but this is already a specific property of QED. Only the electron–positron loop survives.

The surviving loop is a function of $\langle\mathcal{A}\rangle$, and, by derivation, is the electron–positron current averaged over the in-vacuum:

$$\overset{\psi}{\underset{\psi}{\mathcal{A}\!-\!\bigcirc}} = J_\mu^{\text{vac}}(\langle\mathcal{A}\rangle) = \langle\text{in vac}|J_\mu(\hat{\psi})|\text{in vac}\rangle . \tag{4.5}$$

This is the vacuum current. According to (4.3), the *observable* electromagnetic field satisfies the Maxwell equations with an addition of the vacuum current:

$$\nabla^\nu \mathcal{R}_{\nu\mu}(\mathcal{A}) = -J_\mu^{\text{vac}}(\mathcal{A}) - J_\mu^{\text{ext}} . \tag{4.6}$$

We obtain this current by varying the effective action and next replacing the Euclidean resolvents with the retarded resolvents:

$$J_\mu^{\text{vac}}(\mathcal{A}) = \left.\frac{\delta\Gamma(\mathcal{A})}{\delta\mathcal{A}^\mu}\right|_{\Box\to\Box_{\text{ret}}} , \tag{4.7}$$

$$\Gamma(\mathcal{A}) = \int dx \, g^{1/2} \Big[ \mathcal{R} F(\square) \mathcal{R} + F(\square_1, \square_2, \square_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 + \dots \Big] . \qquad (4.8)$$

It is completely similar if $\varphi^a(x)$ is a set of the metric field and any matter fields

$$\text{GRAVITY:} \quad \varphi^a = \Big( g_{\mu\nu}, \psi \Big) . \qquad (4.9)$$

The only difference is that the vertex $S_{ggg}$ is nonvanishing:

$$R_{\mu\nu}(\langle g \rangle) - \frac{1}{2} \langle g_{\mu\nu} \rangle R(\langle g \rangle) + \overset{\psi}{\underset{\psi}{\overset{g}{\bigcirc}}} + \overset{g}{\underset{g}{\overset{g}{\bigcirc}}} = 8\pi T^{\text{ext}}_{\mu\nu} , \qquad (4.10)$$

$$\langle \psi \rangle = 0 , \qquad (4.11)$$

and it is assumed again that the matter fields have no sources. Again, by derivation, the matter loop is the energy-momentum tensor of the field $\hat{\psi}$ averaged over the in-vacuum but the vacuum current contains, in addition, the graviton loop:

$$T^{\text{vac}}_{\mu\nu} == \langle \text{in vac} | T_{\mu\nu}(\hat{\psi}) | \text{in vac} \rangle + \text{ the graviton loop.} \qquad (4.12)$$

The Einstein equations are replaced by the expectation-value equations in the in-vacuum state:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T^{\text{vac}}_{\mu\nu}(g) + 8\pi T^{\text{ext}}_{\mu\nu} . \qquad (4.13)$$

Since the gravitational field couples to everything, the equation (4.10) should contain loops of all matter fields in Nature. The effective actions for all loops including the graviton loop have the same structure:

$$T^{\text{vac}}_{\mu\nu}(g) = -\frac{2}{g^{1/2}} \frac{\delta \Gamma(g)}{\delta g^{\mu\nu}} \Big|_{\square \to \square_{\text{ret}}} , \qquad (4.14)$$

$$\Gamma(g) = \int dx \, g^{1/2} \Big[ R_{..} F(\square) R_{..} + F(\square_1, \square_2, \square_3) R_{1..} R_{2..} R_{3..} + \dots \Big] . \qquad (4.15)$$

Only the coefficients of the formfactors are different. To have the correct coefficients, one would need to know the full spectrum of particles. Therefore, in the case of gravity, the axiomatic approach is most suitable.

Now recall that the curvatures are redundant, and the effective action is in fact a functional of the conserved currents (3.16) and (3.17). Owing to this fact, the expectation-value equations (4.6) and (4.13) close with respect to these currents:

$$\Big( \nabla^\nu \mathcal{R}_{\nu\mu} \Big) + f(\square_{\text{ret}}) \Big( \nabla^\nu \mathcal{R}_{\nu\mu} \Big) + O\Big( \nabla^\nu \mathcal{R}_{\nu\mu} \Big)^2 = -J^{\text{ext}}_\mu , \qquad (4.16)$$

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + f_1(\Box_{\text{ret}})\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$
$$+ f_2(\Box_{\text{ret}})(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)R + O\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)^2 = 8\pi T^{\text{ext}}_{\mu\nu} . \quad (4.17)$$

Of course, with respect to the mean fields, these equations are closed from the outset but, at an intermediate stage, they are closed with respect to the Maxwell and Einstein currents. When solved with respect to these currents, they become literally the Maxwell and Einstein equations with some external sources but *not* the original ones. To make this clear, use the fact that the vacuum terms are proportional to the Planck constant and solve the equations by iteration:

$$\nabla^\nu \mathcal{R}_{\nu\mu} == -J^{\text{ext}}_\mu + f(\Box_{\text{ret}})J^{\text{ext}}_\mu + O\left(J^{\text{ext}}_\mu\right)^2 , \quad (4.18)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T^{\text{ext}}_{\mu\nu} - f_1(\Box_{\text{ret}})8\pi T^{\text{ext}}_{\mu\nu}$$
$$+ f_2(\Box_{\text{ret}})(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)8\pi T^{\text{ext}} + O\left(T^{\text{ext}}_{\mu\nu}\right)^2 . \quad (4.19)$$

These are the Maxwell and Einstein equations with the original sources propagated in a nonlocal and nonlinear manner.

There is an effect in these equations that drives the entire problem.

### Emission of Charges

Consider again QED and suppose that the external source has a compact spatial support. This source is the current of a set of electrically charged particles moving inside a spacetime tube but, since the observable electromagnetic field is the expectation value, only the total current in (4.6) or (4.18) is observable:

$$J^{\text{tot}}_\mu = J^{\text{ext}}_\mu + J^{\text{vac}}_\mu(\mathcal{A}) . \quad (4.20)$$

And the total current has a noncompact spatial support because the vacuum contribution is nonlocal. One may calculate the flux of charge through the support tube of $J^{\text{ext}}$ and even through a wider tube (see Fig. 3), and it will be nonvanishing:

$$e_{\mathcal{T}}(\Sigma_1) - e_{\mathcal{T}}(\Sigma_2) = \frac{1}{4\pi}\int_{\Sigma_1}^{\Sigma_2} J^{\text{vac}}_\mu \, d\mathcal{T}^\mu \neq 0 . \quad (4.21)$$

Here $e_{\mathcal{T}}(\Sigma)$ is the amount of the electric charge contained inside the tube $\mathcal{T}$ at a given instant $\Sigma$. The charge inside the tube is not conserved.

If, when moving away from the support of $J^{\text{ext}}$, the flux (4.21) falls off rapidly, then its nonvanishing only means that the boundary of the original source gets spread. Because of the creation of virtual pairs, this boundary can
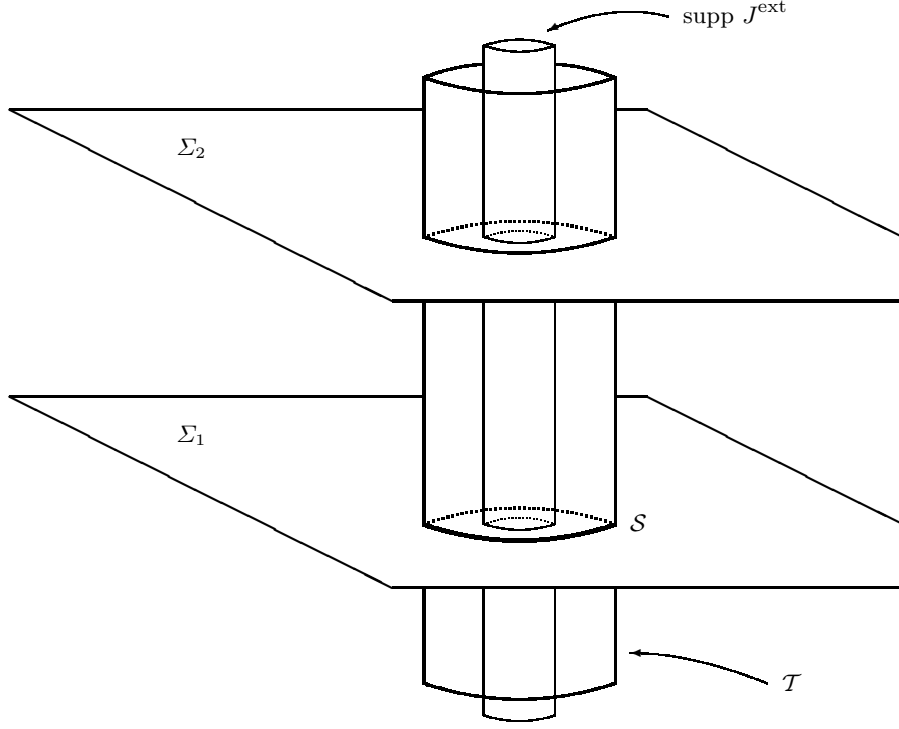
**Fig. 3.** Support tube of $J^{\text{ext}}$ and a wider tube.

never be located precisely. The charges of the external source immersed in the quantum vacuum are always annihilated and created again in a slightly different place. There is no point to worry about. Just step aside a little.

However, one may ask if there is a flux of charge through an infinitely wide tube:

$$e(\Sigma_1) - e(\Sigma_2) = \frac{1}{4\pi} \int\limits_{\Sigma_1}^{\Sigma_2} J_\mu^{\text{vac}}\, d\mathcal{T}^\mu \Big|_{r \to \infty} . \qquad (4.22)$$

In this equation, $e(\Sigma)$ is the total amount of the electric charge in the compact domain of space at a given instant $\Sigma$. For (4.22) to be nonvanishing, $J_\mu^{\text{vac}}$ should behave as

$$J_\mu^{\text{vac}} = O\left(\frac{1}{r^2}\right) , \quad r \to \infty , \qquad (4.23)$$

$$r \propto \sqrt{\text{area of } \mathcal{S}} \qquad (4.24)$$

where $\mathcal{S}$ is the intersection of $\mathcal{T}$ with $\Sigma$ (Fig. 3). In this case, it would turn out that the charge disappears, i.e., *our source is emitting charge*. But even

this may not be a point of concern if the current in (4.22) oscillates with time, and the oscillations sum to zero for a sufficiently long period between $\Sigma_1$ and $\Sigma_2$. The expectation values have uncertainties, and these oscillations are a quantum noise. Just do not measure (4.22) too often.

However, one may ask if the charge emitted for the entire history

$$e(-\infty) - e(+\infty) = \frac{1}{4\pi} \int\limits_{\Sigma\to-\infty}^{\Sigma\to+\infty} J_\mu^{\mathrm{vac}} \, d\mathcal{T}^\mu \Big|_{r\to\infty} \tag{4.25}$$

is nonvanishing. There will always be oscillations in the current but they may sum not to zero. Since, as $r \to \infty$, all fields fall off, there are, in this limit, the asymptotic Killing vectors corresponding to all the symmetries of flat and empty spacetime. Therefore, one may ask the same questions about the emission of energy and any other charges. Thus the quantity

$$M(-\infty) - M(+\infty) = \int\limits_{\Sigma\to-\infty}^{\Sigma\to+\infty} T_{\mu\nu}^{\mathrm{vac}} \xi^\nu \, d\mathcal{T}^\mu \Big|_{r\to\infty} \tag{4.26}$$

with $\xi^\nu$ the asymptotic timelike Killing vector is the energy emitted by the source for the entire history.

If the total emitted charges are nonvanishing, then this is the real effect, and then the question emerges: what are the carriers of these charges? There should be some real agents carrying them away. But the particles of the original source stay in the tube. Besides them, there is only the electron–positron field but it is in the in-vacuum state. This means that, at least initially, there are neither electrons nor positrons. There remains to be assumed a miracle: that either the real electrons or the real positrons – depending on the sign of the emitted charge – get created. Then they are created by pairs, and, say, the created positron is emitted while the created electron stays in the compact domain.

This crazy guess can be checked. We have two ways of calculating the vacuum currents: through the effective action and by a direct averaging of the operator currents as in (4.5) and (4.12). Specifically, for the in-vacuum of electrons and positrons we have

$$T_{\mu\nu}^{\mathrm{vac}} == \langle \mathrm{in\ vac}|T_{\mu\nu}(\hat{\psi})|\mathrm{in\ vac}\rangle \tag{4.27}$$

where $T_{\mu\nu}(\hat{\psi})$ is the operator energy-momentum tensor of the electron–positron field $\hat{\psi}$. The equation for $\hat{\psi}$

$$\left(\slashed{\partial} + \mu - \mathrm{i}q\langle\slashed{A}\rangle\right)\hat{\psi} = 0 \tag{4.28}$$

contains the electromagnetic field which in (4.27) figures as an external field but is in fact the mean field solving the expectation-value equations. We know

that, in the past, all mean fields are static. In the future, they become static again because, if the total emitted charges are finite, then all the processes should die down. Thus, there are two asymptotically static regions: in the past and in the future. The carriers of the emitted charges should be detectable in the future as particles with definite energies. But then the state in which they are absent is the out-vacuum whereas their quantum state is the in-vacuum. *It may be the case that the in-vacuum contains the out-particles.* This will be the case if, between the static regions in the past and future, there is a region where $\langle \mathcal{A} \rangle$ is nonstatic because then the basis functions of the Fock modes that are the eigenfunctions of the energy operator in the future and the basis functions that are such in the past are different solutions of the Dirac equation (4.28).

If we expand $\hat{\psi}$ in the basis solutions of the out-particles, insert this expansion in (4.27), and then insert (4.27) in (4.26), the result will be

$$M(-\infty) - M(+\infty) = \left\langle \text{in vac} \left| \sum_A \varepsilon_A \, \hat{a}_{\text{out}}^{+\,A} \hat{a}_{\text{out}}^{\,A} \right| \text{in vac} \right\rangle \qquad (4.29)$$

where $\varepsilon_A$ is the energy of the out-mode $A$, and similarly for the other charges. This result needs no comments. Miracles happen.

### Emission of Charges (Continued)

An important point concerning miracles is that they happen not always. Let us see what is needed for this particular miracle to happen. For that, it is necessary to introduce characteristic parameters of the problem. There are two sets of parameters.

*Parameters of the quantum field:* $q$, $\mu$.

*Parameters of the external source:* $e$, $l$, $\nu$.

Here, $q$ and $\mu$ are the charge and mass of the vacuum particles (e.g., of the electrons and positrons), $e$ is the charge of the external source, $l$ is the characteristic width of its support tube, and $\nu$ is the frequency parameter that characterizes the nonstationarity of the source.

The vacuum current in (4.18) is of the form

$$J^{\text{vac}} = \int_0^\infty dm^2 \, \rho(m^2) \frac{1}{m^2 - \Box_{\text{ret}}} J^{\text{ext}} + O\left(J^{\text{ext}}\right)^2 . \qquad (4.30)$$

Here and above, the notation $\Box_{\text{ret}}$ is to record that the resolvent is to be taken retarded. The structure of the nonlinear terms in (4.30) is similar: there is an overall resolvent acting on a function quadratic in $J^{\text{ext}}$ (see (2.50)). If the vacuum particles are massive, the spectral weight will be proportional to the $\theta$-function:

$$\rho(m^2) \propto \theta(m^2 - 4\mu^2) \tag{4.31}$$

to tell us that there is a threshold of pair creation. We need to find the behaviour of $J^{\text{vac}}$ at a large distance from the support of $J^{\text{ext}}$:

$$J^{\text{vac}}\Big|_{r \gg l} = ? \tag{4.32}$$

First we need to calculate the action of the retarded resolvent on a source $J^{\text{ext}}$ having a compact spatial support. If $J^{\text{ext}}$ is static, the result is

$$\frac{1}{m^2 - \Box_{\text{ret}}} J^{\text{ext}}\Big|_{r \gg l} = \frac{C}{r} \exp(-mr) , \quad J^{\text{ext}} \text{ static.} \tag{4.33}$$

At a large distance from the source, this is the Yukawa potential. Because the function (4.33) is static, it does not depend on the spacetime direction in which the limit $r \gg l$ is taken. If $J^{\text{ext}}$ is nonstatic, this is no more the case. The limit $r \gg l$ is direction-dependent, and there are directions in which the decrease is slower. Namely, in the directions of the outgoing light rays,

$$\frac{1}{m^2 - \Box_{\text{ret}}} J^{\text{ext}}\Big|_{r \gg l} = \frac{C}{r} \exp\left(-m\sqrt{rU}\right) , \quad J^{\text{ext}} \text{ nonstatic} \tag{4.34}$$

where $U$ is a function of time[5] whose order of magnitude is

$$U \sim \frac{1}{\nu} . \tag{4.35}$$

Expression (4.34) is to be inserted in the spectral integral (4.30), and, since the spectrum is cut off from below, we find that the vacuum current is suppressed by the factor

$$J^{\text{vac}} \sim \exp\left(-\frac{\mu\sqrt{r}}{\sqrt{\nu}}\right) , \quad r \gg l . \tag{4.36}$$

This is what constrains miracles. However, we find also that the suppressing factor depends on the frequency of the source and *can be removed by raising the frequency*. The farther from the support of $J^{\text{ext}}$, the greater the frequency should be for the current to be noticeable. The pair creation starts as soon as the energy $\hbar\nu$ exceeds the threshold

$$\hbar\nu > 2\mu c^2 \tag{4.37}$$

but, for the source to emit charge, the frequency should be even greater:

$$\hbar\nu > (\mu c^2)\left(\frac{\mu c}{\hbar}l\right) . \tag{4.38}$$

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[5] Of the retarded time since the surfaces $\Sigma$ to which the outgoing light rays belong are null.

This is easy to understand. The particles start being created in the support of the source with small momenta and cannot go far away. The extra factor $(\mu c/\hbar)l$ in (4.38) may be interpreted as the number of created particles for which there is room in the support of the source. If the creation is more violent, the particles get out of the tube. This is the meaning of condition (4.38). The mechanism of emission and conservation of charge is illustrated in Fig. 4. There are initially the charges of the external source in its support tube. They repel the like particles of the created pairs and, when the number of the latter exceeds $(\mu c/\hbar)l$, push them out of the tube. The unlike particles stay in the tube and diminish its charge.
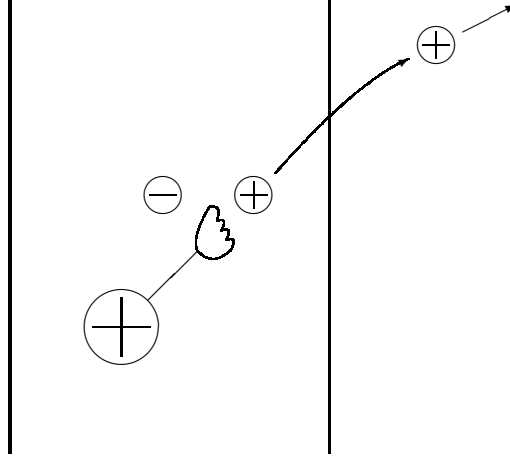


**Fig. 4.** Mechanism of emission and conservation of charge.

Since the cause of the vacuum instability is the nonstationarity of the external source, it is interesting to consider the case where the energy $\hbar\nu$ exceeds overwhelmingly all the other energy parameters of the problem. One can then study the strong effect of particle production. It is assumed, in particular, that $\hbar\nu$ exceeds both the rest energy of the vacuum particle and its Coulomb energy in the external field:

$$\hbar\nu \gg \mu c^2 \, , \tag{4.39}$$

$$\hbar\nu \gg \frac{qe}{l} \, . \tag{4.40}$$

In the limit (4.39), the flux of charge at a given distance from the source ceases depending on the mass $\mu$, and the vacuum particles can be considered as massless. Condition (4.40) enables one to get rid of the consideration of the static vacuum polarization which is irrelevant to the problem. The approximation (4.39) and (4.40) is called high-frequency approximation.

The effective action has been calculated above as an expansion in powers of the curvature but the conditions of validity of this expansion have not been discussed. This lack can now be met. It is the high-frequency approximation in which this expansion is valid. Indeed, consider the series (4.8). Every next term in this series contains an extra power of $\mathcal{R}$, and, by dimension, its formfactor contains an extra power of $\Box^{-1}$. The commutator curvature is proportional to the charges and to $\hbar^{-1}$:

$$\mathcal{R} \sim \frac{qe}{\hbar l^2} \ . \tag{4.41}$$

In the limit $r \gg l$ along the outgoing light rays, the operator $\Box$ contains one time derivative:

$$\Box \sim \frac{\nu}{l} \ . \tag{4.42}$$

As a result, every next term of the series contains, as compared to the previous one, the extra factor

$$\frac{qe}{\hbar \nu l} \ll 1 \ . \tag{4.43}$$

In addition, the formfactors in (4.8) can be calculated in the massless limit, as has been done above.

However, the inquest of miracles is not yet completed. Assuming that the vacuum particles are massless or that the high-frequency regime holds, we get rid of the suppressing exponential in (4.36) but we still need to check the power of decrease of the current. The power should be the one in (4.23) for the emission of charge to occur. We can readily check this since we know the behaviour of the resolvent. Expression (4.34) is again to be inserted in the spectral integral (4.30) but this time assuming that the spectrum begins with zero mass:

$$J^{\text{vac}}\Big|_{r\gg l} = \int\limits_{0}^{\infty} dm^2 \, \rho(m^2) \frac{C}{r} \exp\left(-m\sqrt{rU}\right) \ . \tag{4.44}$$

We see that, for the current to decrease as $O(1/r^2)$, the spectral weight should have a finite and nonvanishing limit at zero mass:

$$\rho(0) = \text{finite} \neq 0 \ . \tag{4.45}$$

For the respective formfactor, this is a condition on its behaviour at small $\Box$. The behaviour should be

$$F(\Box) = \int\limits_{0}^{\infty} dm^2 \, \frac{\rho(m^2)}{m^2 - \Box} \quad \xrightarrow[\Box \to 0]{} \quad - \rho(0) \ln(-\Box) \ . \tag{4.46}$$

We arrive at the following consistency condition on the vacuum formfactors. In the limit where one (any) of the $\Box$ arguments is small and the others are fixed, the formfactors should not grow faster than $\ln(-\Box)$:

$$F(\Box)\Big|_{\Box \to 0} = \text{const.} \ln(-\Box) \ , \tag{4.47}$$

$$F(\Box_1, \Box_2, \Box_3)\Big|_{\Box_1 \to 0} = f(\Box_2, \Box_3)\ln(-\Box_1) \,, \tag{4.48}$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

If they grow faster, the charges cannot be maintained finite, i.e., an isolated system cannot exist in such a vacuum. If they grow as $\ln(-\Box)$, the theory of isolated systems is consistent but these systems emit charges. If they grow slower, the charges are conserved.

One can check whether the one-loop formfactors satisfy this consistency condition. The second-order formfactors (3.90)–(3.94) do. The third-order formfactors behave generally as [35]

$$F(\Box_1, \Box_2, \Box_3)\Big|_{\Box_1 \to 0} = f(\Box_2, \Box_3)\frac{1}{\Box_1} + g(\Box_2, \Box_3)\ln(-\Box_1) + \ldots \,. \tag{4.49}$$

The alarming terms $1/\Box$ appear only in the arguments acting on the gravitational curvatures. Therefore, they can affect only the vacuum energy-momentum tensor, and it has been checked that, in the energy-momentum tensor, these terms coming from different formfactors cancel. In the currents, the one-loop formfactors satisfy strictly the consistency condition. Since, in addition, their asymptotic $\ln(-\Box)$ terms are nonvanishing, the emission of charges in the high-frequency regime is real. The only thing that remains to be checked is that this emission is not a pure quantum noise. It will be checked by a direct calculation.

Now one can answer also the question about the indefinite local terms in the effective action. The coefficients of these terms are the unspecified constants in (3.90)–(3.94). In the limit $\Box \to 0$, the values of these constants are immaterial. Only the terms $\ln(-\Box)$, $\Box \to 0$ of the formfactors work, and, therefore, the incompleteness of local quantum field theory does not affect the presently considered problem.

It will be noted that there are now two mechanisms by which an isolated system can emit energy. One is purely classical: a nonstationary source can emit the electromagnetic or gravitational waves. The other is quantum: immersed in the vacuum, a nonstationary source can emit also charged particles. A high-frequency source will generally emit *both*.

## Particle Creation by External Fields

The problem of particle creation by external fields is a part of the expectation-value problem. In the context of the foregoing, it can be set as follows. Consider the quantum field that satisfies a linear second-order equation

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu\hat{1} + \hat{P}\right)\phi = 0 \tag{4.50}$$

containing three external fields: the metric, the connection, and the potential. The external fields are asymptotically static in the past and future but otherwise arbitrary except that their currents

$$J_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \ , \tag{4.51}$$

$$\hat{J}_\alpha = \nabla^\beta \hat{\mathcal{R}}_{\alpha\beta} \ , \tag{4.52}$$

$$\hat{Q} = \hat{P} + \frac{1}{6}R\hat{1} \tag{4.53}$$

are confined to a spacetime tube. The quantum field is in the in-vacuum state. What is the energy of the quanta of the field $\phi$ created by the external fields for the entire history? In the high-frequency approximation, we have everything to answer this question.

To formulate the answer, I need some preliminary construction. Every current has an associated quantity called its radiation moment. It will now be defined.

Consider a timelike geodesic in the external metric of equation (4.50). It enters the domain of nonstationarity of external fields with a definite energy and goes out of this domain with a definite energy. Let $E$ be its energy per unit rest mass on going out. I am only interested in the geodesics that escape to $r = \infty$. They have $E > 1$, and, instead of $E$, I shall use the parameter $\gamma$ defined as

$$\gamma = \frac{\sqrt{E^2 - 1}}{E} \ , \quad E > 1 \ , \quad 0 < \gamma < 1 \ . \tag{4.54}$$

At $r = \infty$, the geodesic has a certain spatial direction, or, equivalently, it comes to a certain point of the celestial 2-sphere. I shall denote this sphere as $\mathcal{S}$, its points as $\theta$:

$$\theta = (\theta_1, \theta_2) \ , \quad \theta \in \mathcal{S} \ , \tag{4.55}$$

and the integral over the unit 2-sphere as

$$\int d^2\mathcal{S}(\theta)\,(\cdots) \ . \tag{4.56}$$

A geodesic with given $\gamma$ and $\theta$ will be called $\gamma, \theta$ -geodesic (see Fig. 5).

A $\gamma, \theta$ -geodesic can be emitted from every point of a compact domain. Therefore, the $\gamma, \theta$ -geodesics with *the same values* of $\gamma$ and $\theta$ make a congruence, and it can be proven that this congruence is hypersurface-orthogonal. Let the orthogonal hypersurfaces be

$$T_{\gamma\theta}(x) = \text{const.} \tag{4.57}$$

Since the parameters $\gamma, \theta$ fix the congruence, they fix also the family of the orthogonal hypersurfaces (4.57), and the "const." in (4.57) fixes a member of the family. The function $T_{\gamma\theta}$ is determined up to a transformation $T_{\gamma\theta} \to f(T_{\gamma\theta})$. This arbitrariness will be removed by the normalization condition

$$(\nabla T_{\gamma\theta})^2 = -\left(1 - \gamma^2\right) \tag{4.58}$$

and the condition that the vector $\nabla T_{\gamma\theta}$ is past directed. It is a property of the geodetic congruences that the norm in (4.58) can be chosen constant.
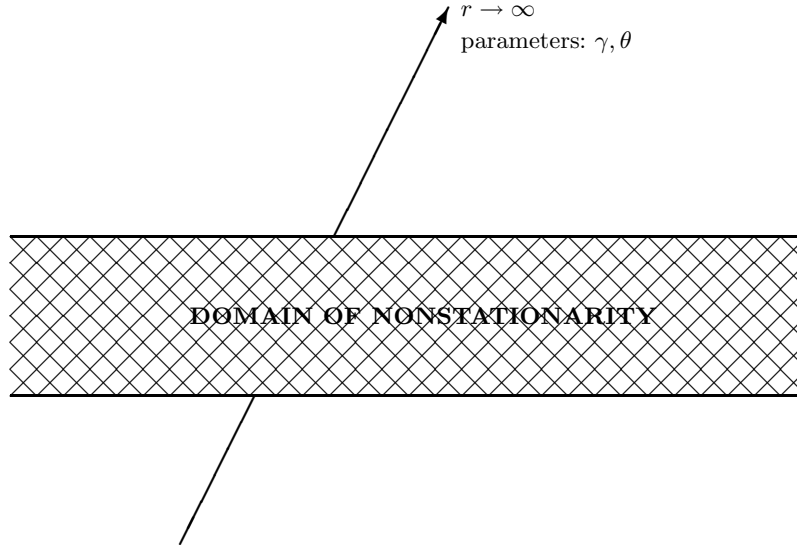
**Fig. 5.** A $\gamma, \theta$ -geodesic.

The radiation moment of any scalar current $J$ is the following hypersurface integral:

$$D = \frac{1}{4\pi} \int dx \, g^{1/2} \delta \left( T_{\gamma\theta}(x) - \tau \right) J(x) \ . \tag{4.59}$$

If the current is not a scalar, it should first be parallel transported from the integration point to $r = \infty$ along the respective $\gamma, \theta$ -geodesic. Thus if the current is a vector, its radiation moment is

$$D^\alpha = \frac{1}{4\pi} \int dx \, g^{1/2} \delta \left( T_{\gamma\theta}(x) - \tau \right) J^\beta(x) a_\beta{}^\alpha(x, \infty) \tag{4.60}$$

where $a_\beta{}^\alpha(x, \infty)$ is the propagator of parallel transport of vectors to infinity along the $\gamma, \theta$ -geodesic emanating from $x$. The radiation moment $D^\alpha$ is then a vector at infinity. In the same way, the radiation moment is defined for any current. For the three currents (4.51)–(4.53), the radiation moments will be denoted respectively as

$$J_{\alpha\beta}, \ \hat{J}_\alpha, \ \hat{Q} \longrightarrow D_{\alpha\beta}, \ \hat{D}_\alpha, \ \hat{D} \ . \tag{4.61}$$

Since the indices of the radiation moments pertain to a point at infinity, their contractions like

$$\hat{D}_\alpha \hat{D}^\alpha = g_{\alpha\beta} \hat{D}^\alpha \hat{D}^\beta \ , \quad \text{etc.} \tag{4.62}$$

always assume the flat metric $g_{\alpha\beta}$ at infinity. All radiation moments are functions of four parameters:

$$D = D(\gamma, \theta, \tau) \ . \tag{4.63}$$

In the limit $\gamma = 1$, the $\gamma, \theta$ -geodesics become null. The orthogonal hyper-surfaces (4.57) also become null, and the geodesics themselves become their generators. For the radiation moments, this is a regular limit. Nothing special happens to them in this limit except that they become very important. The radiation moments at $\gamma = 1$ govern the emission of waves in classical theory. Thus if $J_\alpha$ in (4.52) is an electric current, then the following expression:

$$\left(M(-\infty) - M(+\infty)\right)_{\text{electromagnetic waves}}$$
$$= \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} d\tau \int d^2\mathcal{S}(\theta) \left[g_{\alpha\beta}\left(\frac{d}{d\tau}D^\alpha\right)\left(\frac{d}{d\tau}D^\beta\right)\right]\Bigg|_{\gamma=1} \tag{4.64}$$

is the energy of the electromagnetic waves emitted by this current for the entire history. A similar expression with the tensor current (4.51):

$$\left(M(-\infty) - M(+\infty)\right)_{\text{gravitational waves}}$$
$$= \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} d\tau \int d^2\mathcal{S}(\theta) \frac{1}{2}(g_{\alpha\mu}g_{\beta\nu} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu})\left(\frac{d}{d\tau}D^{\alpha\beta}\right)\left(\frac{d}{d\tau}D^{\mu\nu}\right)\Bigg|_{\gamma=1}$$
$$\tag{4.65}$$

is the energy of the gravitational waves emitted by the current $J_{\alpha\beta}$ for the entire history.

The radiation moment is a generating function for the multipole moments. The multipole expansion is the expansion of $D$ at $\gamma = 0$. It makes sense for nonrelativistic systems since $\gamma$ is proportional to $1/c$.

Expressions (4.64) and (4.65) are the solutions of the classical radiation problem. And here is the solution of the quantum radiation problem [50]:

$$\left(M(-\infty) - M(+\infty)\right)_{\text{created particles}}$$
$$= \frac{1}{(4\pi)^2} \int\limits_0^1 d\gamma\, \gamma^2 \int\limits_{-\infty}^{\infty} d\tau \int d^2\mathcal{S}(\theta)\, \mathrm{tr}\left[\left(\frac{d^2}{d\tau^2}\hat{D}\right)^2\right.$$
$$- \frac{1}{3}\frac{1}{(1-\gamma^2)}g_{\alpha\beta}\left(\frac{d}{d\tau}\hat{D}^\alpha\right)\left(\frac{d}{d\tau}\hat{D}^\beta\right)$$
$$+ \left.\frac{1}{30}\hat{1}(g_{\alpha\mu}g_{\beta\nu} - \frac{1}{3}g_{\alpha\beta}g_{\mu\nu})\left(\frac{d^2}{d\tau^2}D^{\alpha\beta}\right)\left(\frac{d^2}{d\tau^2}D^{\mu\nu}\right)\right].$$
$$\tag{4.66}$$

This is the energy of the quanta of the field $\phi$ created by the external fields for the entire history. As compared to the expressions above, there is an extra time derivative in the case of the tensor and scalar moments. It accounts for

the dimension of the coupling constant. Also, instead of setting $\gamma = 1$, one needs to integrate over $\gamma$. Otherwise, the similarity is striking. The quantum problem of particle creation becomes almost the same thing as the classical problem of emission of waves.

The presence in (4.66) of an integral over $\gamma$ is not just a technical detail. The radiation moments have both the longitudinal projections, i.e., the projections on the direction of the geodesic at infinity and the transverse projections. Inspecting the contractions of the moments in (4.64)–(4.66), one can see that, at $\gamma = 1$, the longitudinal projections drop out of these contractions. In the integral over $\gamma$, also the longitudinal projections survive. Owing to this fact, spherically symmetric sources cannot emit waves but can produce particles from the vacuum.

Now I can explain why, when expanding the effective action, I stopped at the terms cubic in the curvature. In the high-frequency approximation, the expansion (3.40) needs to be calculated up to the lowest-order terms that give a nonvanishing effect. The terms of first order in the curvature are local and give no effect. The terms of second order in the curvature are nonlocal and contribute to the energy flux at infinity but it turns out that *their contribution is a pure quantum noise*. The real effect of particle production begins with the third order in the curvature. Expression (4.66) results from the triangular loop diagrams.

Since varying the action destroys one curvature, a cubic action generates a quadratic current. This gives the radiation energy a chance to be positive definite. Expression (4.66) is positive definite indeed:

$$\Big(M(-\infty) - M(+\infty)\Big)_{\text{created particles}} \geq 0 . \tag{4.67}$$

In particular, for the matrix contributions, this follows from relations (3.11), (3.12) and the positive definiteness of the matrix $\omega_{ab}$:

$$\text{tr}\left(\frac{d^2}{d\tau^2}\hat{D}\right)^2 \geq 0 , \quad \text{tr}\left[g_{\alpha\beta}\left(\frac{d}{d\tau}\hat{D}^\alpha\right)\left(\frac{d}{d\tau}\hat{D}^\beta\right)\right] \leq 0 . \tag{4.68}$$

The positivity of the gravitational-field contribution can be proven directly.

### The Backreaction Problem

The energy emitted by an isolated system (in all forms) should be bounded both from below and from above: it should be positive and less than the energy stored in the initial state

$$0 \leq \Big(M(-\infty) - M(+\infty)\Big) \leq M(-\infty) . \tag{4.69}$$

In expression (4.66), the positivity is guaranteed but the energy conservation is not. The reason is that the setting of the problem with external fields is

physically inconsistent. The vacuum current determines the solution of the mean-field equations, and the mean field rather than the external field determines the vacuum current. If the backreaction of the vacuum is neglected, the conservation laws need not be observed.

One case in which the vacuum backreaction may not be neglected is where both mechanisms of the energy emission, classical and quantum, are engaged simultaneously. This concerns particularly the vector connection field. In expression (4.66), the integral over $\gamma$ has a pole $(1-\gamma)^{-1}$ in the term with the vector moment. The residue of the integrand in this pole is precisely the quantity (4.64), i.e., the energy of the outgoing waves of the vector connection field. If it is nonvanishing, e.g., if the external source emits both the electromagnetic waves and the electrically charged particles, the integral in $\gamma$ diverges. The result is a disaster: the radiation energy appears to be infinite. In fact it should be taken into account that the created charge affects the generation of the electromagnetic waves, and the respective changes in the electromagnetic field affect the creation of charge. In the self-consistent solution, the disaster is removed.

Another example concerns the metric field when it has an event horizon. In this case, the integral in $\tau$ diverges at the upper limit. By construction, $\tau$ is the time of an external observer. As $\tau \to \infty$, the source moving in the tube hits the event horizon. Its proper time does not turn into infinity. The integrand in (4.66) is just finite in this limit, and the integral in $\tau$ diverges linearly. This is the Hawking constant flux of radiation from the black hole. If its backreaction on the metric is neglected, the total emitted energy is infinite.

But even when the quantity (4.66) is finite, it depends on the frequency of the source. If the source is external, this frequency is a free parameter. The energy of created quanta grows with frequency, and, typically, the ratio

$$\frac{M(-\infty) - M(+\infty)}{M(-\infty)}\bigg|_{\nu \to \infty} \sim \ln \nu \qquad (4.70)$$

also grows so that, at a sufficiently high frequency, the energy conservation law will be violated. The backreaction should take into account that, when the source creates real particles, it loses energy and slows down. It then creates less particles, and the process dies away. The conservation laws will then be restored.

The backreaction problem has been solved only in a few cases [51]–[56]. The examples for which it has been solved show that the solution can be unexpected and interesting.

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